Uniformization of discrete Riemann surfaces

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On the basis of the notion of discrete conformal equivalence of Euclidean triangle meshes we define discrete conformal equivalence of spherical and hyperbolic triangulations [1, 3, 4]. We consider triangulated surfaces equipped with a metric of constant curvature $K = 0,\ 1,$ or -1 except at the vertices of the triangulation, where the metric is allowed to have cone-like singularities. The discrete metric of such a surface is the function that assignes to each geodesic edge ij its length l_{ij}. Now let $l : E \to \mathbb{R}_{>0}$ be the discrete metric of a triangulated Euclidean surface and let $\tilde{l} : E \to \mathbb{R}_{>0}$ be the discrete metric of an (a) Euclidean, (b) hyperbolic, or (c) spherical surface with combinatorially equivalent triangulation. Then l and $\tilde{l}$ are called *discretely conformally equivalent* if there exists a function $u : V \to \mathbb{R}$ on vertices such that

$$\text{(a) } \tilde{l}_{ij} = e^{\frac{1}{2}(u_i+u_j)} l_{ij}, \quad \text{(b) } 2\sinh\frac{\tilde{l}_{ij}}{2} = e^{\frac{1}{2}(u_i+u_j)} l_{ij}, \quad \text{(c) } 2\sin\frac{\tilde{l}_{ij}}{2} = e^{\frac{1}{2}(u_i+u_j)} l_{ij}.$$

A discrete Riemann surface is an equivalence class of discretely conformally equivalent metrics. It is characterized by the length-cross-ratios defined on edges

$$\mathrm{lcr}_{ij} = \frac{l_{ik} l_{jl}}{l_{kj} l_{li}},$$

where k and l are the vertices of two triangles sharing the edge ij.

The discrete uniformization problem is formulated as follows: Given a discrete metric, find a discretely conformally equivalent Euclidean, spherical, or hyperbolic

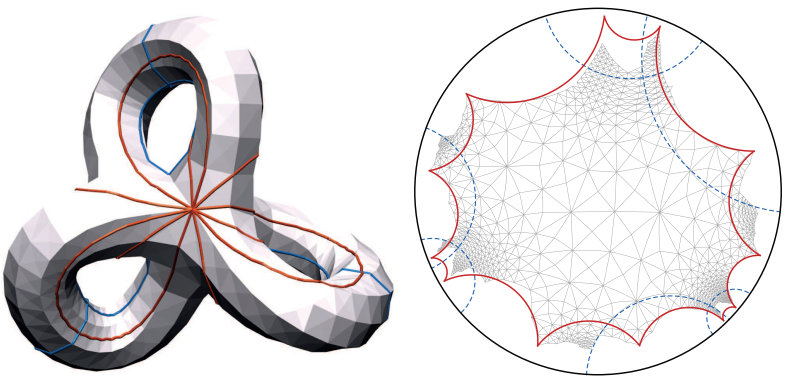

Figure 1. An embedded genus 3 surface and its uniformization. The dashed lines are the axes of the hyperbolic translations that identify corresponding edges of the fundamental polygon. The curves on the embedded surface are the pre-images of the polygon and its axes.

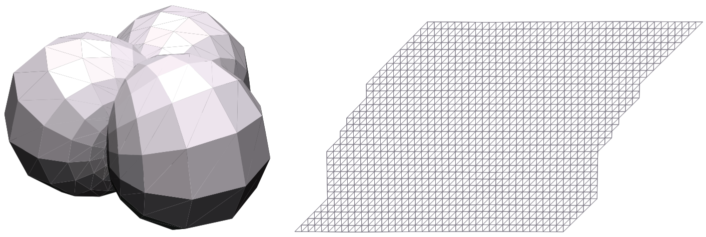

Figure 2. Uniformization of a discretely sampled conformal immersion of the Wente torus. The faces of the polyhedral surface are approximate conformal squares. In the discrete uniformization their images are approximately squares, as it should be.

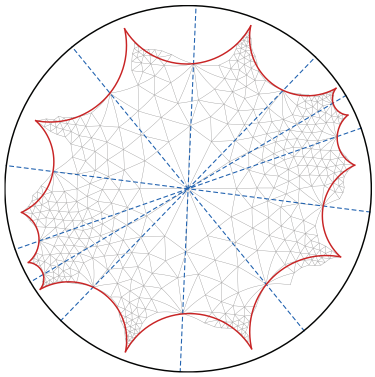

Figure 3. Uniformization of a genus 3 hyperelliptic surface created from a two sheeted branched cover of the Riemann sphere. The dashed lines are the axes of the hyperbolic translations that identify opposite sides of the fundamental polygon.

metric without cone-like singularities at vertices, i.e., such that the sum of angles of corresponding triangles around every vertex is 2π. As in the smooth case: For triangulated surfaces of genus $g = 0,\ 1,$ or >1 one obtains a Euclidean, spherical, or hyperbolic discretely conformally equivalent metric, respectively. In all three cases we give a variational description of the corresponding uniformization problem. It is related to volumes of ideal hyperbolic polyhedra [1, 5]. In the Euclean and hyperbolic case the corresponding functional is convex. Using this technique we show how to calculate standard representations of discrete Riemann surfaces (Figures 1, 2).

For higher genus surfaces we calculate Fuchsian uniformization groups and show different examples for hyperelliptic and general surfaces (Figure 1). We derive a hyperellipticity criterion from the Fuchsian group representation. If and only if the surface is hyperelliptic then in a normalized presentation where opposite sides

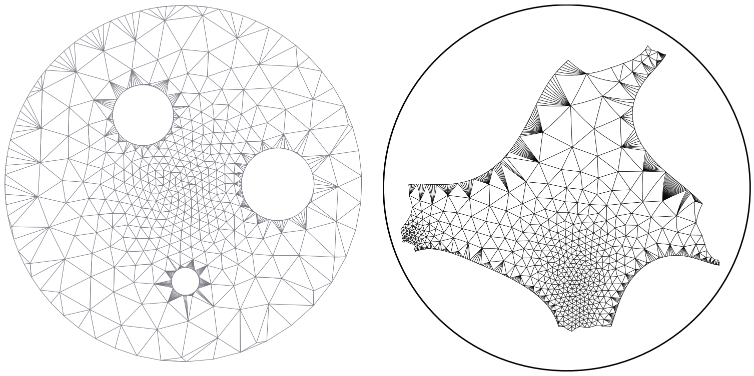

FIGURE 4. Fuchsian uniformization of a discrete Riemann surface given by Schottky data. The fundamental domain is bounded by images of the Schottky-circles and cuts that connect them.

of a fundamental polygon are identified, the axes of the hyperbolic translations meet in a point (Figure 3).

We show how to pass from a Schottky to a Fuchsian uniformization. Here one cannot start with the edge length of a triangulation of a Schottky fundamental domain, since the edges of identified boundaries have different lengths. But the length-cross-ratios lcr : $E \to \mathbb{R}_{>0}$ are well defined. They determine a discrete conformal class of globally defined discrete metrics l that can be used to obtain a Fuchsian uniformization. The Schottky-circles are mapped to smooth curves in the corresponding Fuchsian uniformization (Figure 4).

References

[1] A. Bobenko, U. Pinkall, B. Springborn, *Discrete conformal maps and ideal hyperbolic polyhedra.* arXiv:1005.2698 (May 2010).

[2] A. I.. Bobenko, S. Sechelmann, B. Springborn, *Uniformization of discrete Riemann surfaces.* In preparation.

[3] F. Luo, *Combinatorial Yamabe flow on surfaces.* Commun. Contemp. Math. **6** (2004), 765–780.

[4] B. Springborn, P. Schröder, U. Pinkall, *Conformal Equivalence of Triangle Meshes.* ACM Transactions on Graphics **27**, Proceedings of ACM SIGGRAPH 2008.

[5] I. Rivin, *Euclidean structures on simplicial surfaces and hyperbolic volume.* Ann. of Math. **139** (1994), 553–580.

Geometric properties of anti-de Sitter simplices and applications

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(joint work with Jeffrey Danciger and Sara Maloni)

Ideal hyperbolic polyhedra have interesting properties that come up in different areas of mathematics. They are uniquely determined by either their dihedral