Uniformization of discrete Riemann surfaces

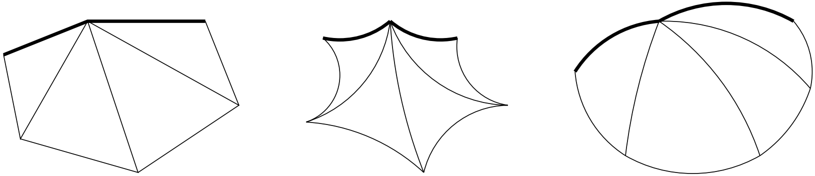
Stefan Sechelmann

Institut für Mathematik, TU-Berlin

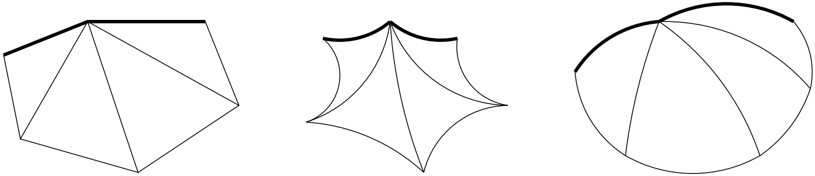
joint work with
Alexander I. Bobenko and Boris Springborn

DFG Research Center Matheon
SFB/TR 109: Discretization in Geometry and Dynamics

A discrete surface is a collection of triangles equipped with a metric of constant curvature. They are glued along geodesic edges. Vertices can have cone-like singularities.

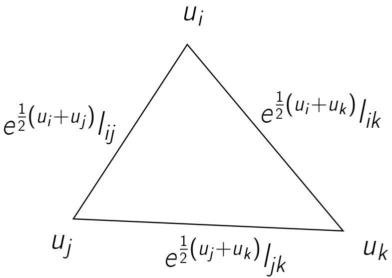


Geodesic edge lengths are called a discrete Euclidean (K=0), hyperbolic (K=-1), or spherical (K=1) metric.



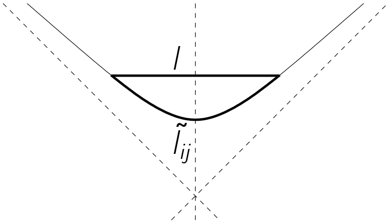
A discrete Euclidean metric with edge length l_{ij} is discretely conformally equivalent to the discrete Euclidean metric $\tilde{l}_{ij}$ if the is a function $u : V \to \mathbb{R}$ such that for all edges

$$\tilde{l}_{ij} = e^{\frac{1}{2}(u_i+u_j)} l_{ij}$$

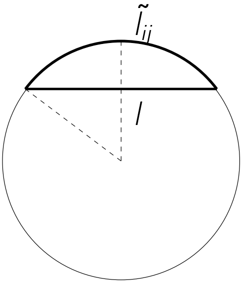


A discrete Euclidean metric l and a discrete (hyperbolic, spherical) metric $\tilde{l}$ are discretely conformally equivalent if

$$2\sinh\frac{\tilde{l}_{ij}}{2} = l_{ij} \qquad\qquad 2\sin\frac{\tilde{l}_{ij}}{2} = l_{ij}$$



hyperbolic



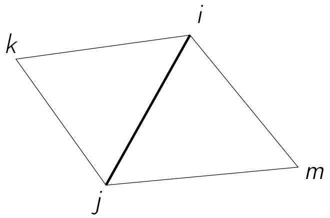
spherical

Definition

A *discrete Riemann surface* is an equivalence class of discretely conformally equivalent metrics.

The Euclidean conformal invariant is the length cross-ratio defined on edges. Two discrete Euclidean metrics are equivalent if their length cross-ratios coincide.

$$\mathrm{lcr}_{ij} = \frac{l_{ik} l_{jm}}{l_{mi} l_{kj}}$$



Uniformization Problem

Given a discrete Riemann surface, find a metric of constant curvature without cone singularities.

As in the smooth case:

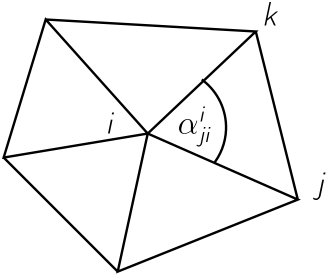
- $g = 1 \rightsquigarrow$ Euclidean
- $g > 1 \rightsquigarrow$ hyperbolic
- $g = 0 \rightsquigarrow$ spherical

Variational Description

Metrics without cone-like singularities are critical points of Euclidean, hyperbolic, and spherical functionals. Angles are calculated in the respctive geometry.

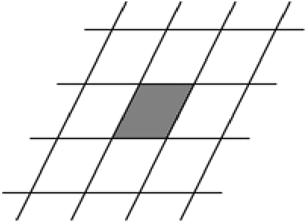
$$\frac{\partial E}{\partial u_i} = 2\pi - \sum_{ijk \ni i} \alpha^i_{jk}$$

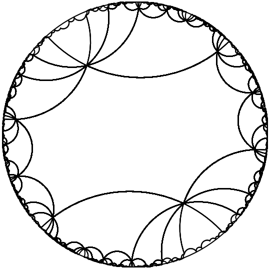
Optimize the corresponding functional to solve the uniformization problem.

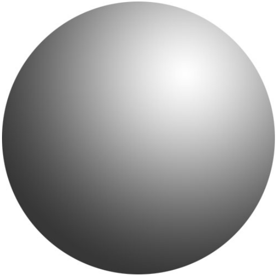


Realizations

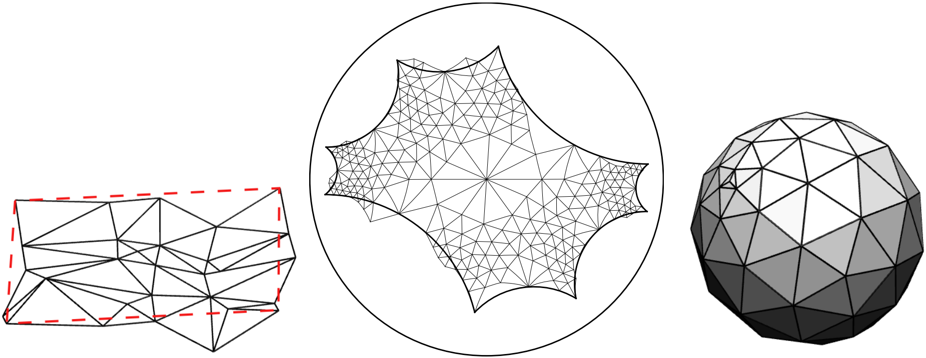
- $g = 1$ - Euclidean plane and lattice Λ, $\mathbb{E}/\Lambda$
- $g > 1$ - Hyperbolic plane and Fuchsian group G, $\mathbb{H}/G$
- $g = 0$ - Sphere







Discrete Realizations



Realizations of discrete metrics without cone-like singularities

Examples

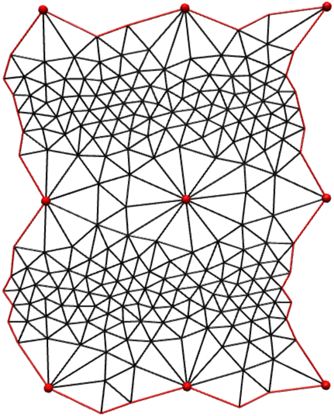
- Fuchsian uniformization of elliptic and hyperelliptic surfaces given as two-sheeted cover of $\hat{\mathbb{C}}$
- Fuchsian uniformization of Schottky data
- Surfaces with boundary

- Algebraic description

$$w^2 = \prod_{i=1}^{2g+2} (z - \lambda_i)$$

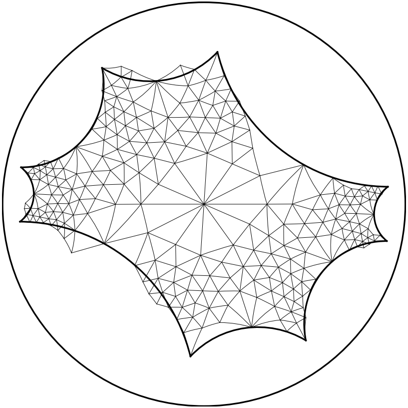
- Riemann surface is a two-sheeted cover of $\hat{\mathbb{C}}$ with branch points λ_i
- Use spherical triangulation with $2g + 2$ singularities at λ_i with cone angle 4π.
- Find conformally equivalent hyperbolic metric without cone-like singularities
- Realization as $\mathbb{E}/T$ or $\mathbb{H}/G$

branched cover



Riemann surface of genus 1 given by a branched cover of $\hat{\mathbb{C}}$

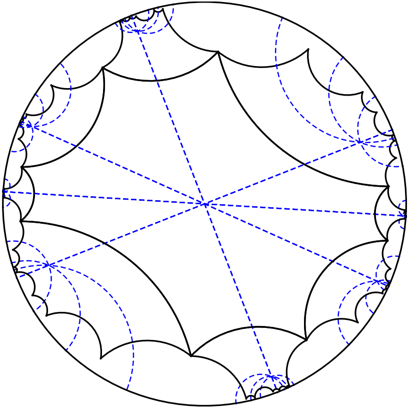
branched cover



Riemann surface of genus 2 given by a branched cover of $\hat{\mathbb{C}}$

Theorem

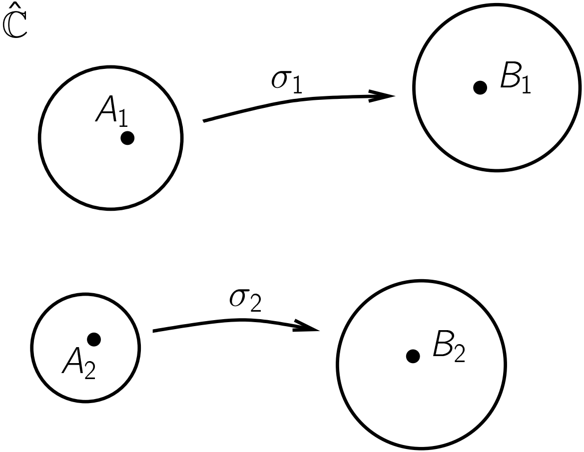
A Riemann surface is hyperelliptic if and only if the axes of the hyperbolic motions that identify opposite sides of a fundamental polygon meet in a point.



Uniformization of Schottky data

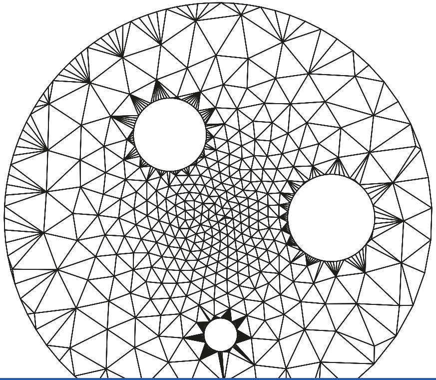
$$\frac{\sigma(z) - A}{\sigma(z) - B} = \lambda \frac{z - A}{z - B}$$

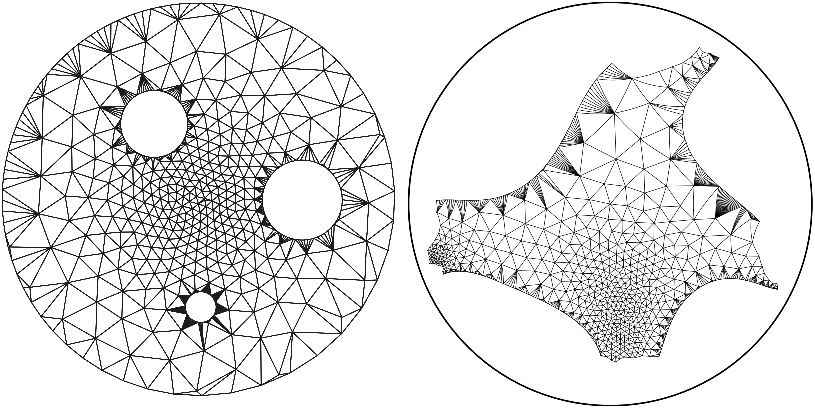
$$|\lambda| < 1$$



Uniformization of Schottky data

- Create triangulation of a fundamental domain with matching vertices on circles
- find length-cross-ratios on circle edges
- pick a metric from the conformal class
- find conformally equivalent hyperbolic metric without cone-like singularities

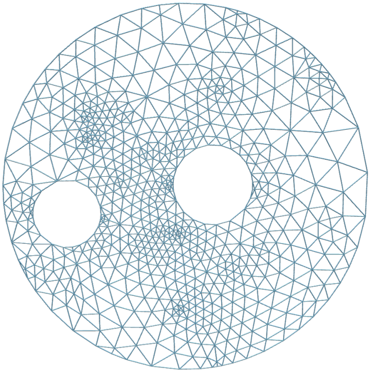




Surface of genus 2 given as $\hat{\mathbb{C}}/G$ and Fuchsian uniformization.

Map boundary components to circles in $\mathbb{E}$, $\mathbb{H}$, or $\mathbb{S}$.

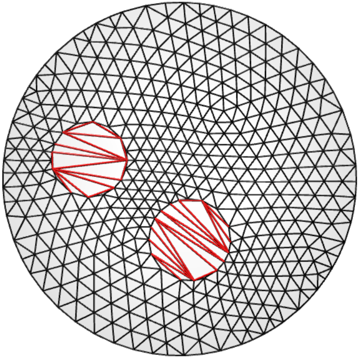
sphere 3 holes



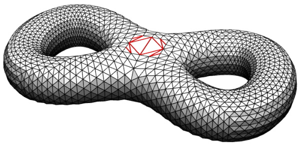
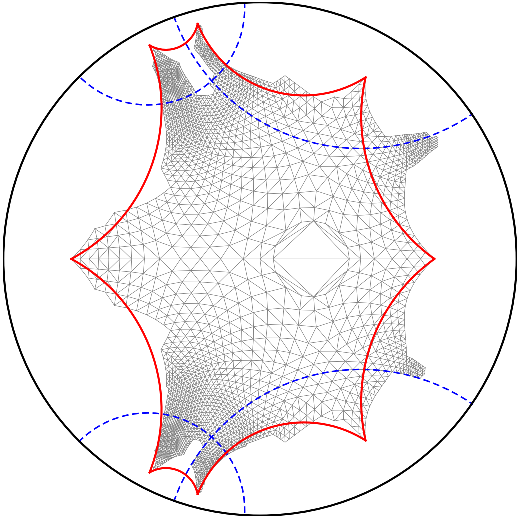
Genus 0 surface with 3 boundary components and uniformization

If logarithmic edge lengths λ are variables of the functional then

$$\frac{\partial E}{\partial \lambda_{ij}} = \pi - \alpha_{ij}^{k} - \alpha_{ji}^{m}$$



Hyperbolic surface with boundary

Genus 2 Riemann surface with one boundary component and Fuchsian uniformization