Uniformization of Elliptic and Hyperelliptic Curves via Discrete Conformal Equivalence

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joint work with A. Bobenko and B. Springborn

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SFB TRR 109 | Discretization in Geometry and Dynamics

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* Hyperelliptic Curves
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* Elliptic Curves
    * Convergence
    * Elliptic Functions

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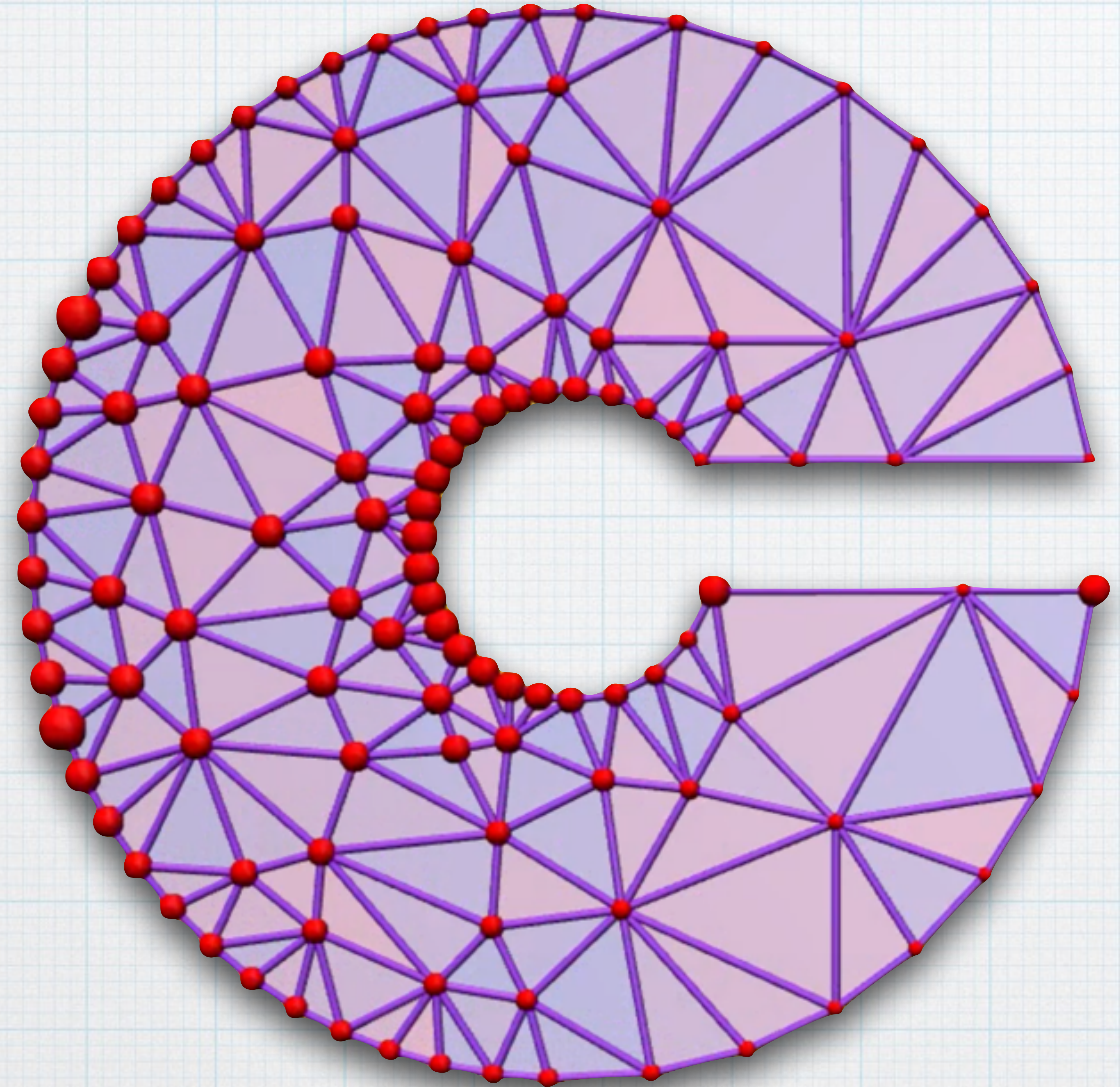
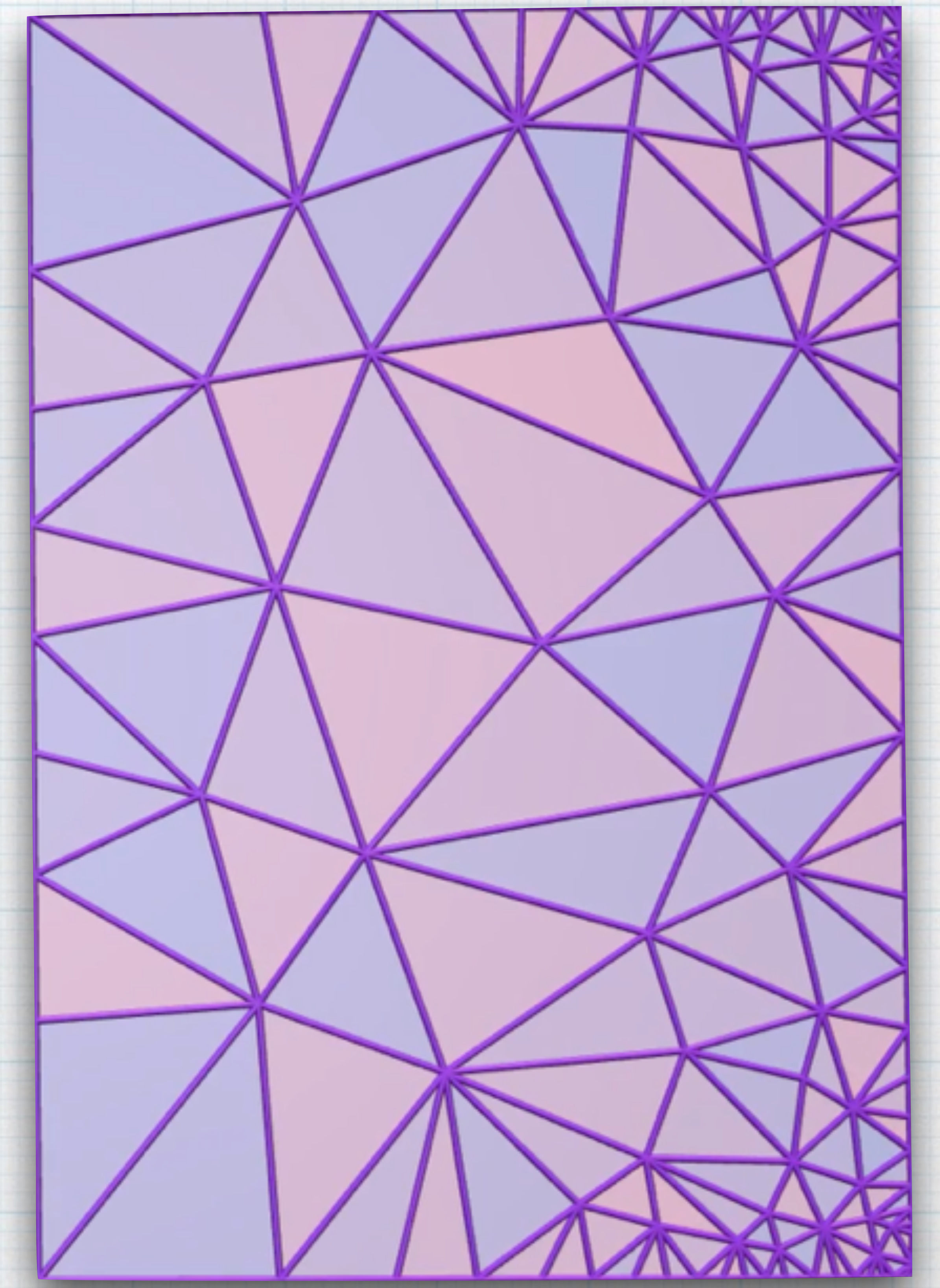
Discrete Conformal Equivalence





$$\tilde{\ell}_{ij} = \mu_i \mu_j \ell_{ij}$$

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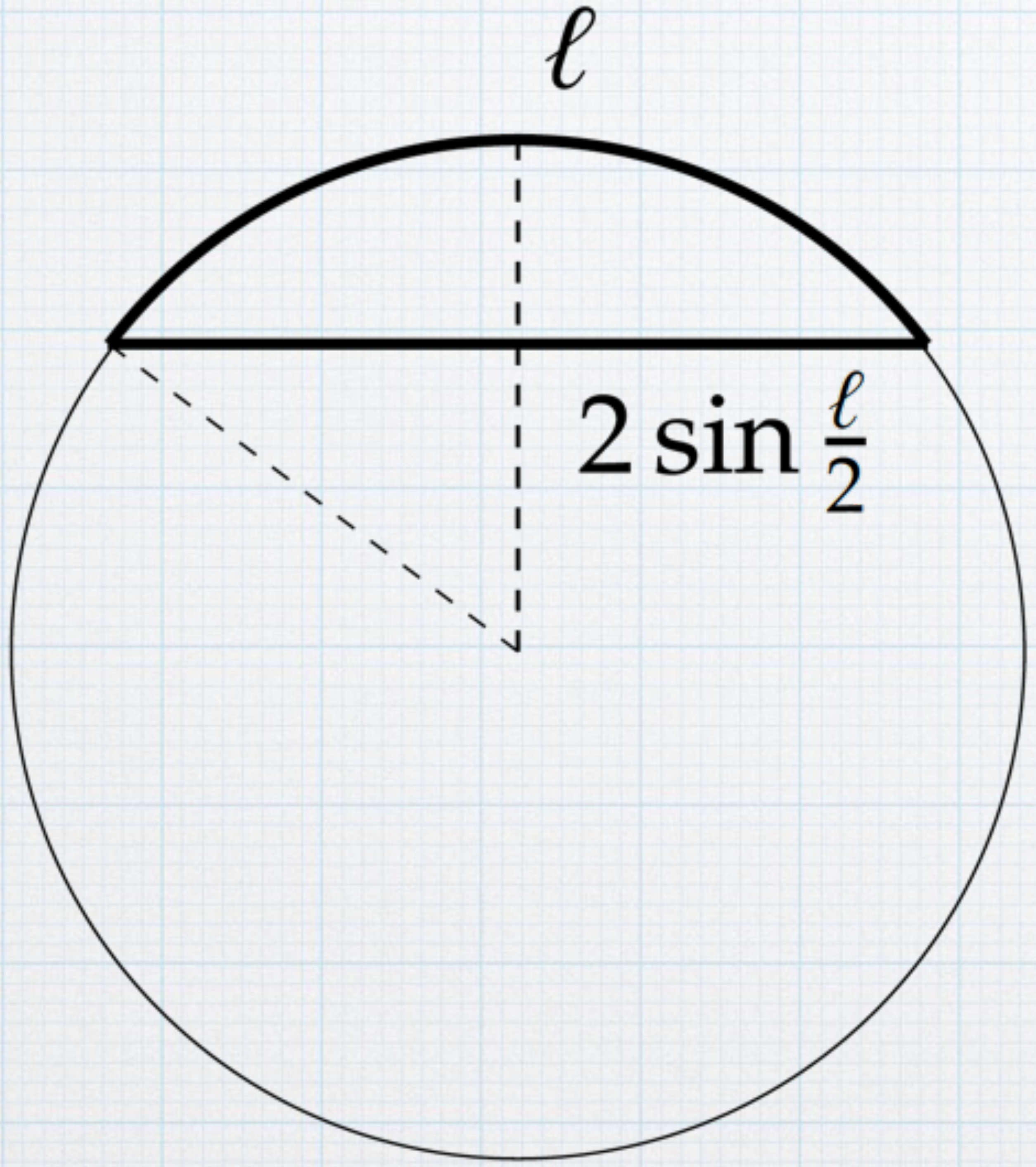
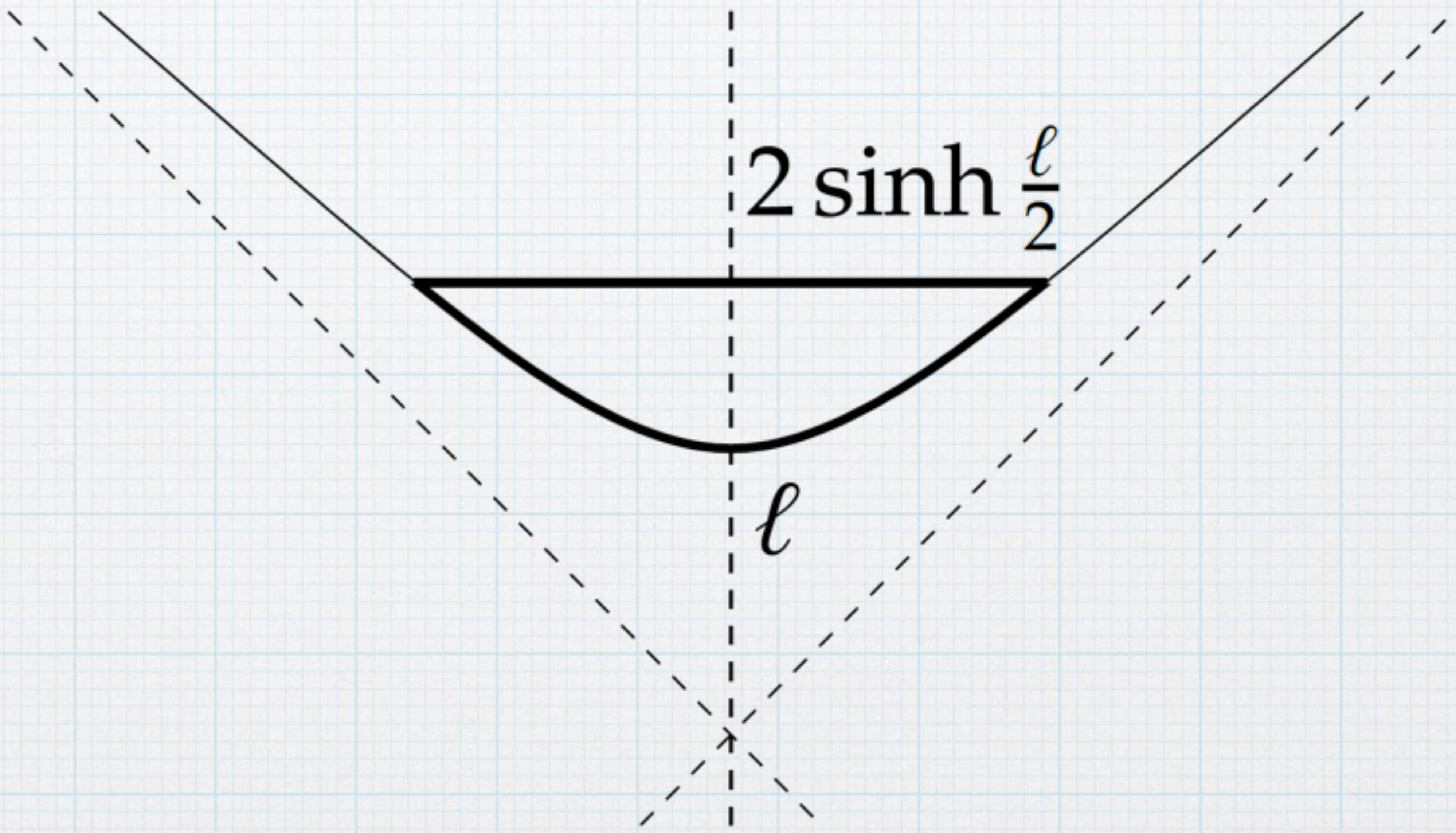
Equivalence of Euclidean and Spherical/Hyperbolic Metrics





$$\sin\left(\frac{\tilde{\ell}_{ij}}{2}\right) = \mu_i\mu_j\ell_{ij}$$

$$\sinh\left(\frac{\tilde{\ell}_{ij}}{2}\right) = \mu_i\mu_j\ell_{ij}$$

Mapping Problem

Given Euclidean lengths ℓ_{ij} find new euclidean/hyperbolic/spherical lengths $\tilde{\ell}_{ij}$ such that

$$\tilde{\ell}_{ij} = \mu_i \mu_j \ell_{ij} \qquad \sin\left(\frac{\tilde{\ell}_{ij}}{2}\right) = \mu_i \mu_j \ell_{ij} \qquad \sinh\left(\frac{\tilde{\ell}_{ij}}{2}\right) = \mu_i \mu_j \ell_{ij}$$

$$\sum_{ijk \ni i} \tilde{\alpha}^i_{jk} = \Theta_i$$

Formulas for α depend on the geometry.

Variational Principle

$$\tilde{\ell}_{ij} = e^{u_i + u_j} \ell_{ij}$$

$$E^{\text{euc}}, E^{\text{hyp}}, E^{\text{sph}} : \mathbb{R}^V \longrightarrow \mathbb{R},$$
$$u \longmapsto E^{\tilde{g}}(u)$$

$$\frac{\partial E^{\tilde{g}}}{\partial u_i}(u) = \Theta_i - \sum_{ijk \ni i} \tilde{\alpha}^i_{jk}$$

$$D^2 E^{\tilde{g}}(u) = \frac{1}{2} \sum_{ijk \in F} \left(q^k_{ij}(u) + q^i_{jk}(u) + q^j_{ki}(u) \right)$$

Solved via convex optimization

Hyperelliptic Curves

$$\left\{ (\mu, \lambda) \in \mathbb{C}^2 \mid \mu^2 = \prod_{k=1}^{2g+2} (\lambda - \lambda_k) \right\}$$

- 1 dimensional complex submanifold of $\mathbb{C}^2$
- Conformally equivalent to a 2-sheeted branched cover of $\hat{\mathbb{C}}$ with branch points at $\lambda_1, \ldots, \lambda_{2g+2}$
- A surface that can be realized as 2-sheeted branched cover of $\hat{\mathbb{C}}$ is called *hyperelliptic surface*

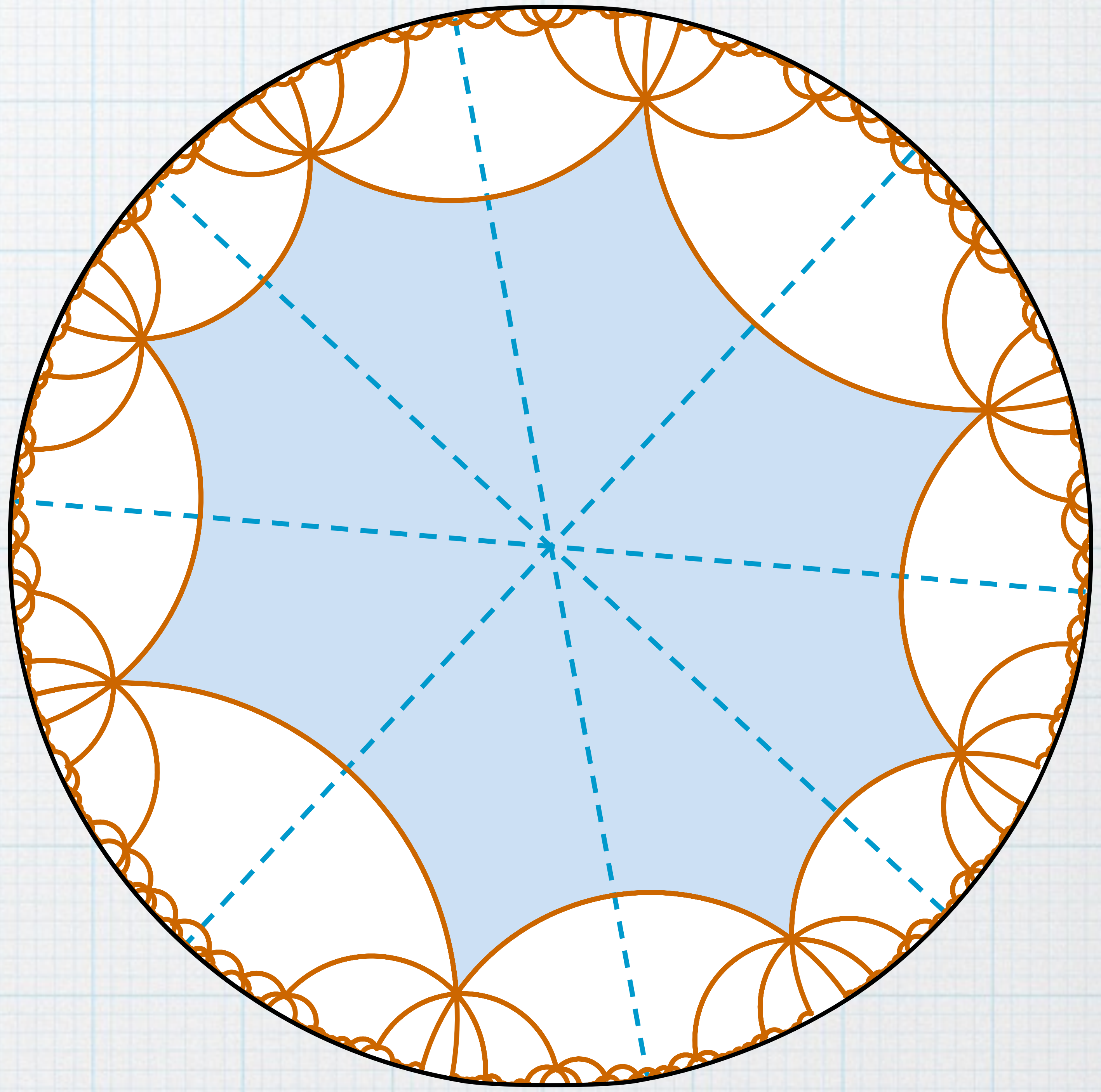
Characterization

Theorem 1 (Schmutz Schaller 1999) *Let R be a closed hyperbolic surface of genus g. Then the following statements are equivalent:*

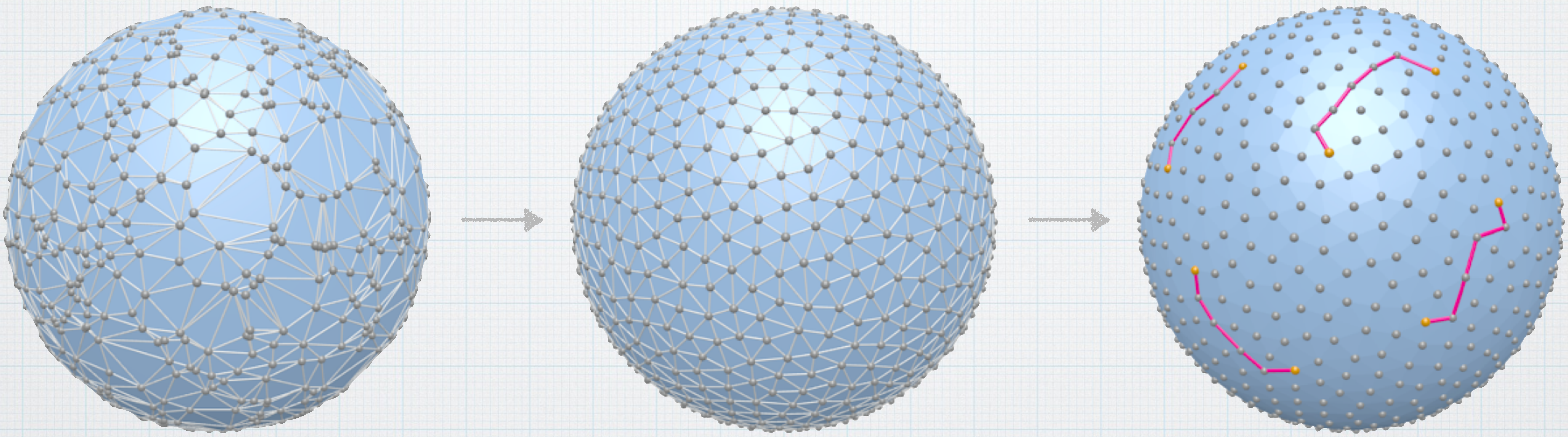
(i) *R is hyperelliptic.*

(ii) *R has a set of $2g$ simple closed geodesics which all intersect in one point and which intersect in no other point.*

(iii) *R has a fundamental polygon that is a $4g$-gon with opposite sides identified and equal opposite angles.*



Discretization



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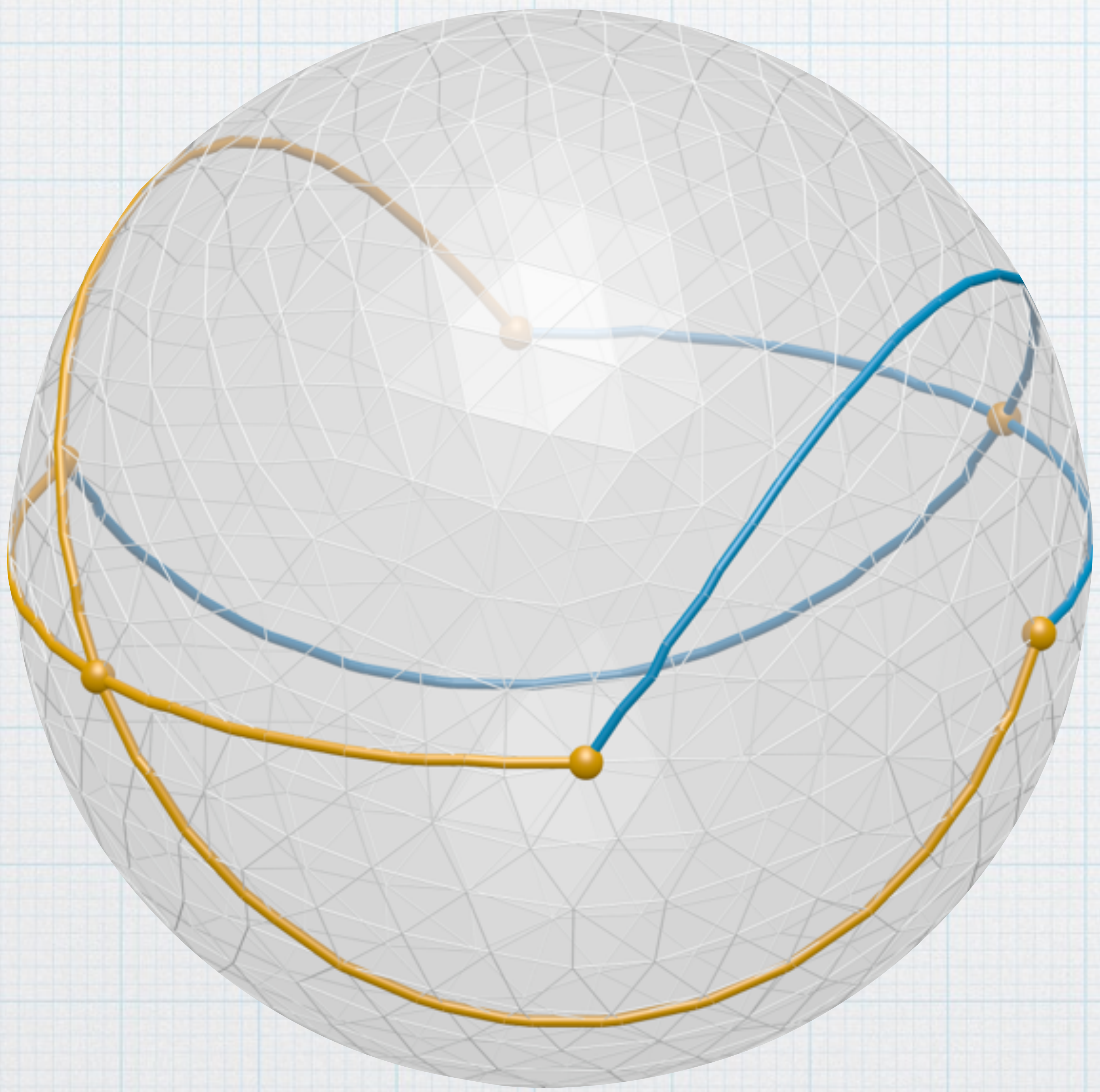
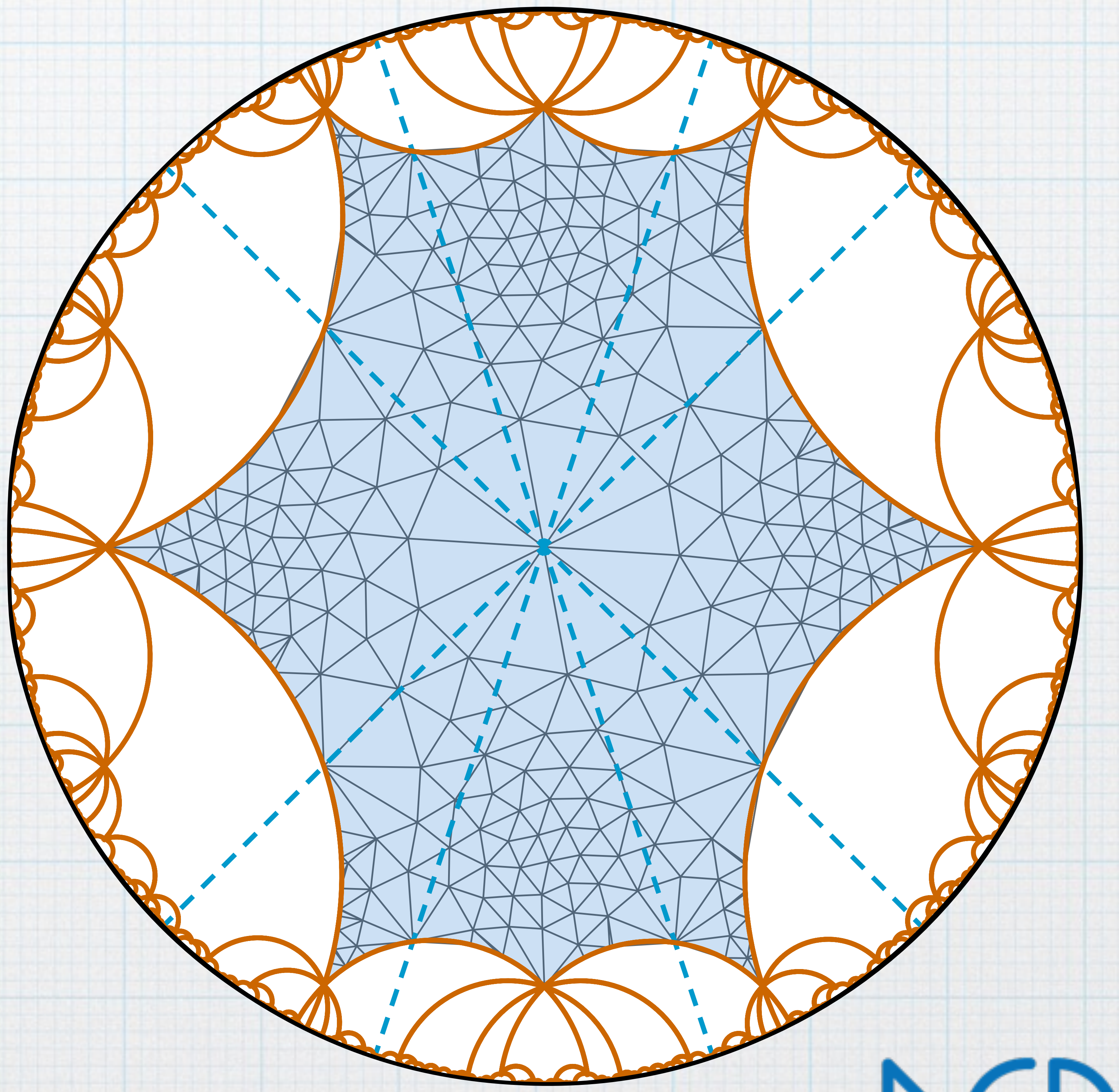
Example

$$\mu^2 = \prod_{k=1}^{6} \left(\lambda - e^{\frac{ik\pi}{3}} \right)$$





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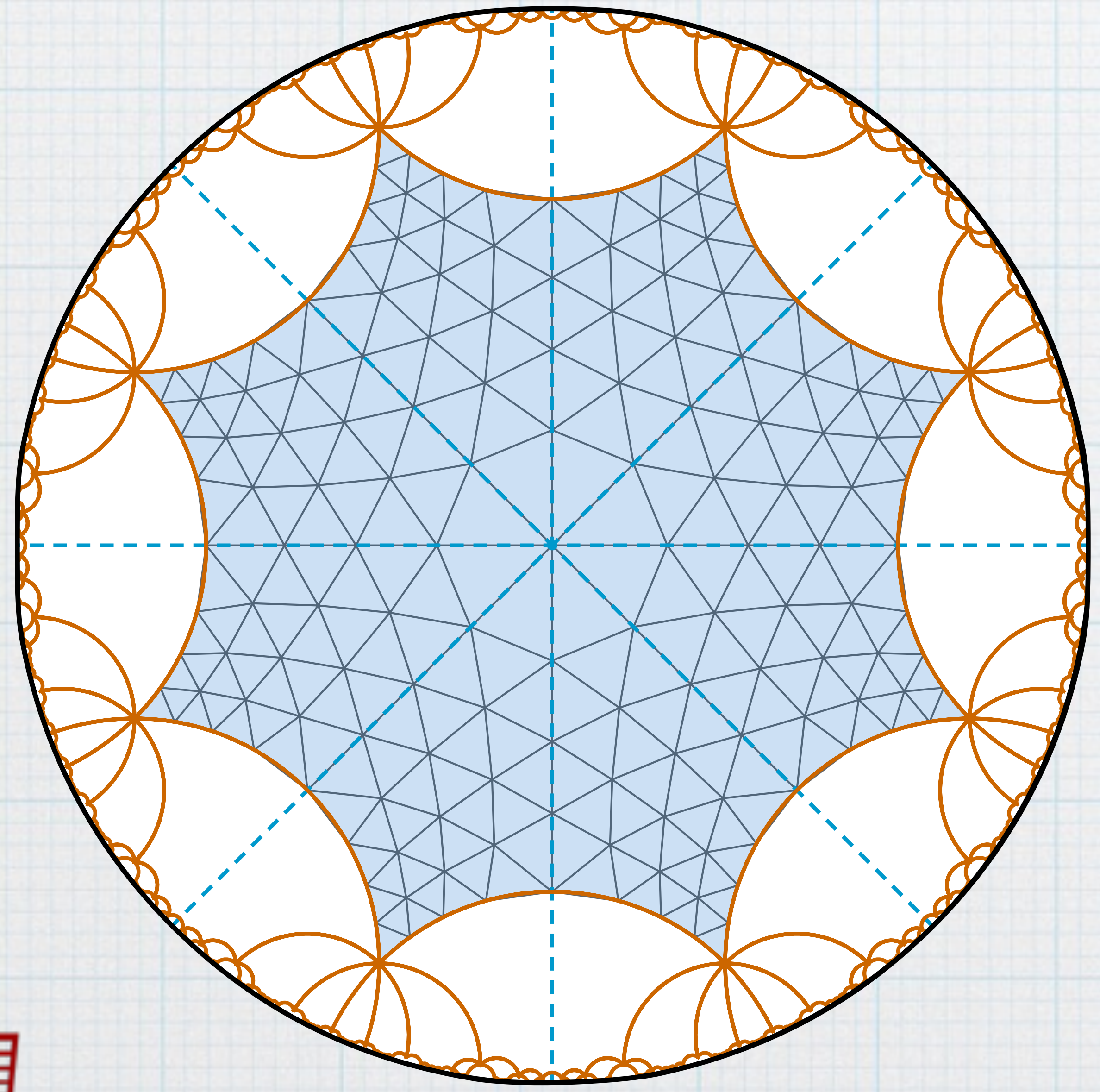
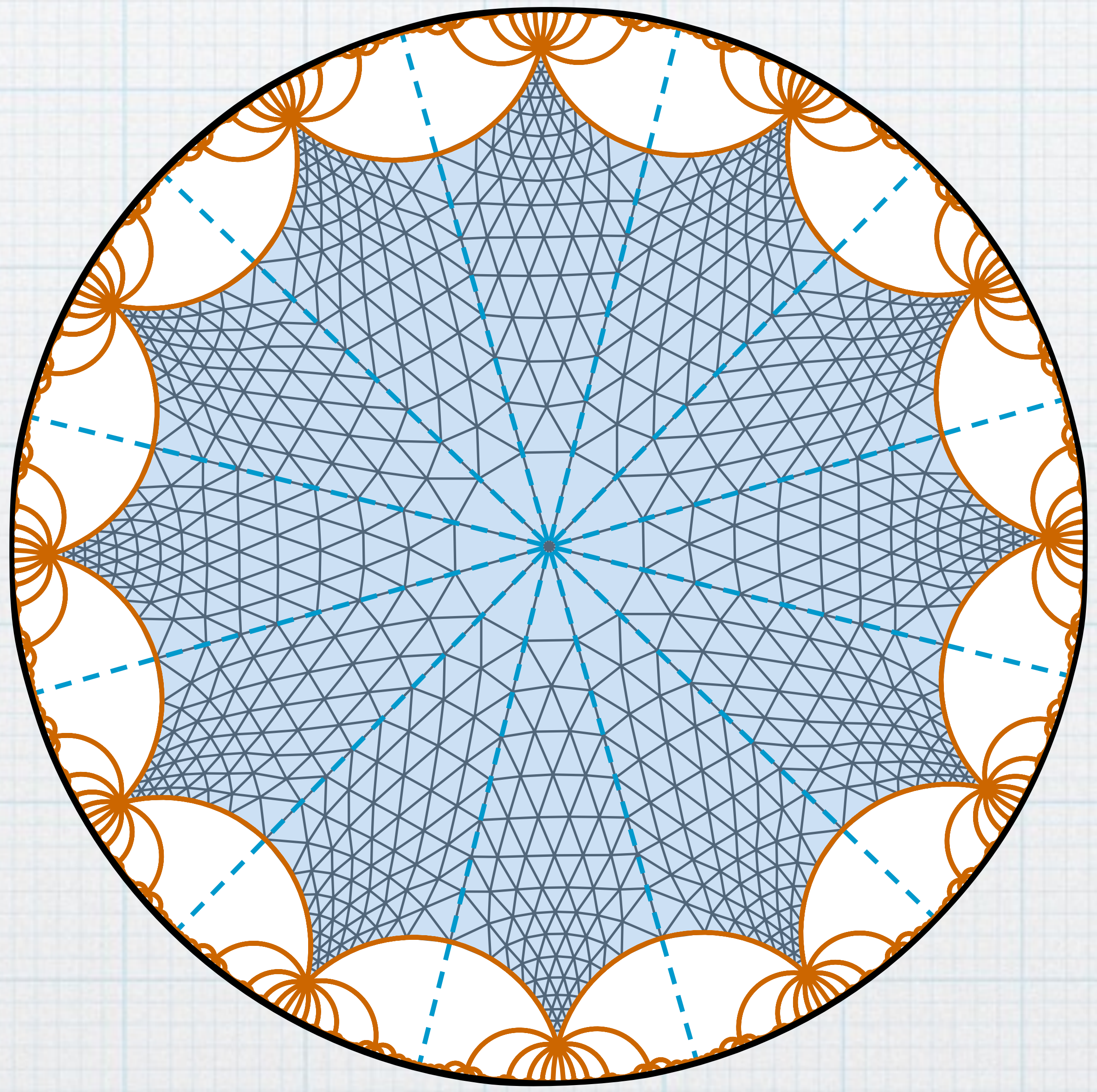
Example: Regular Domain

$$\mu^2 = \lambda \prod_{k=1}^{2g} \left(\lambda - e^{\frac{ik\pi}{g}}\right)$$

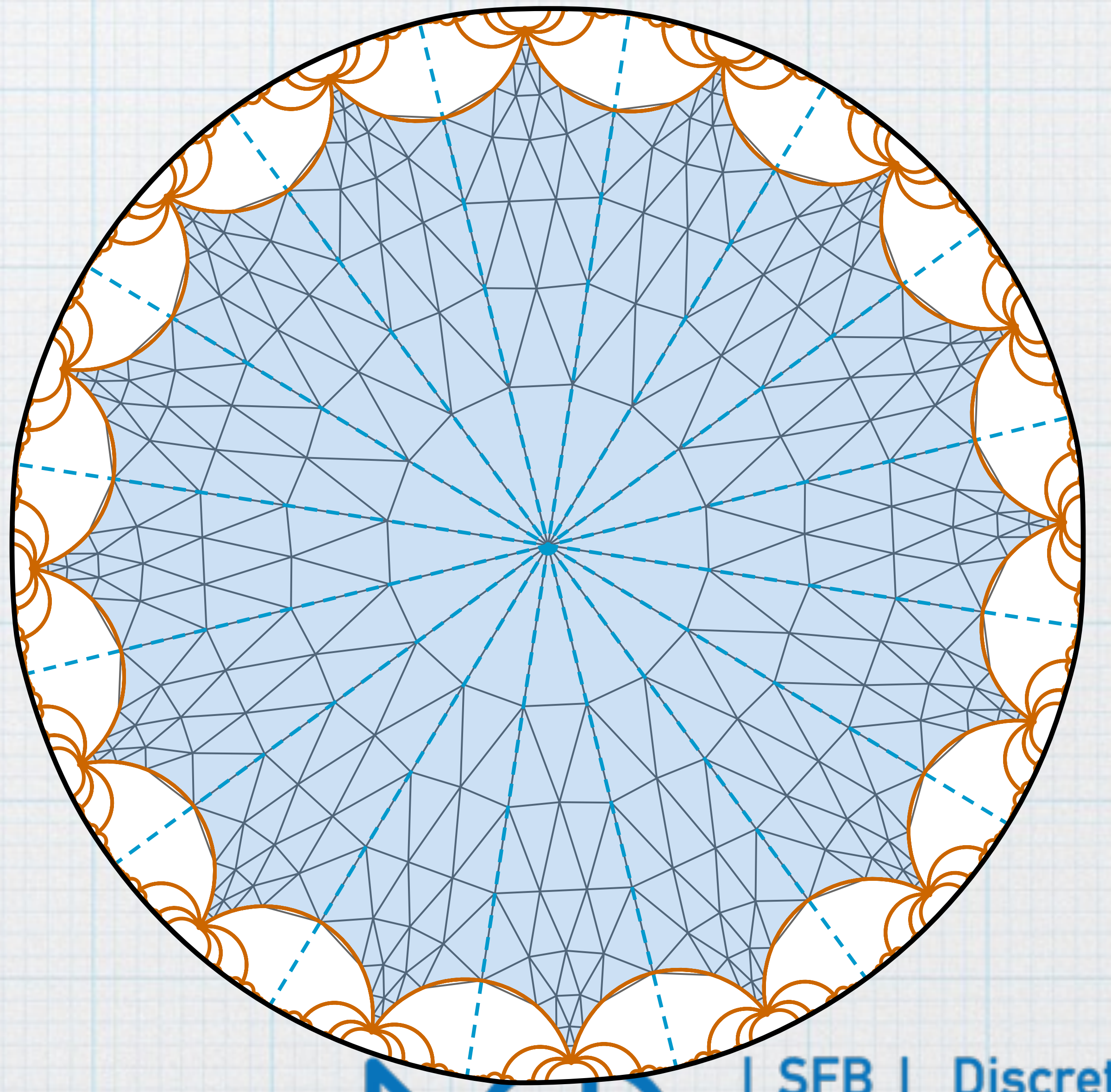
$g = 2$



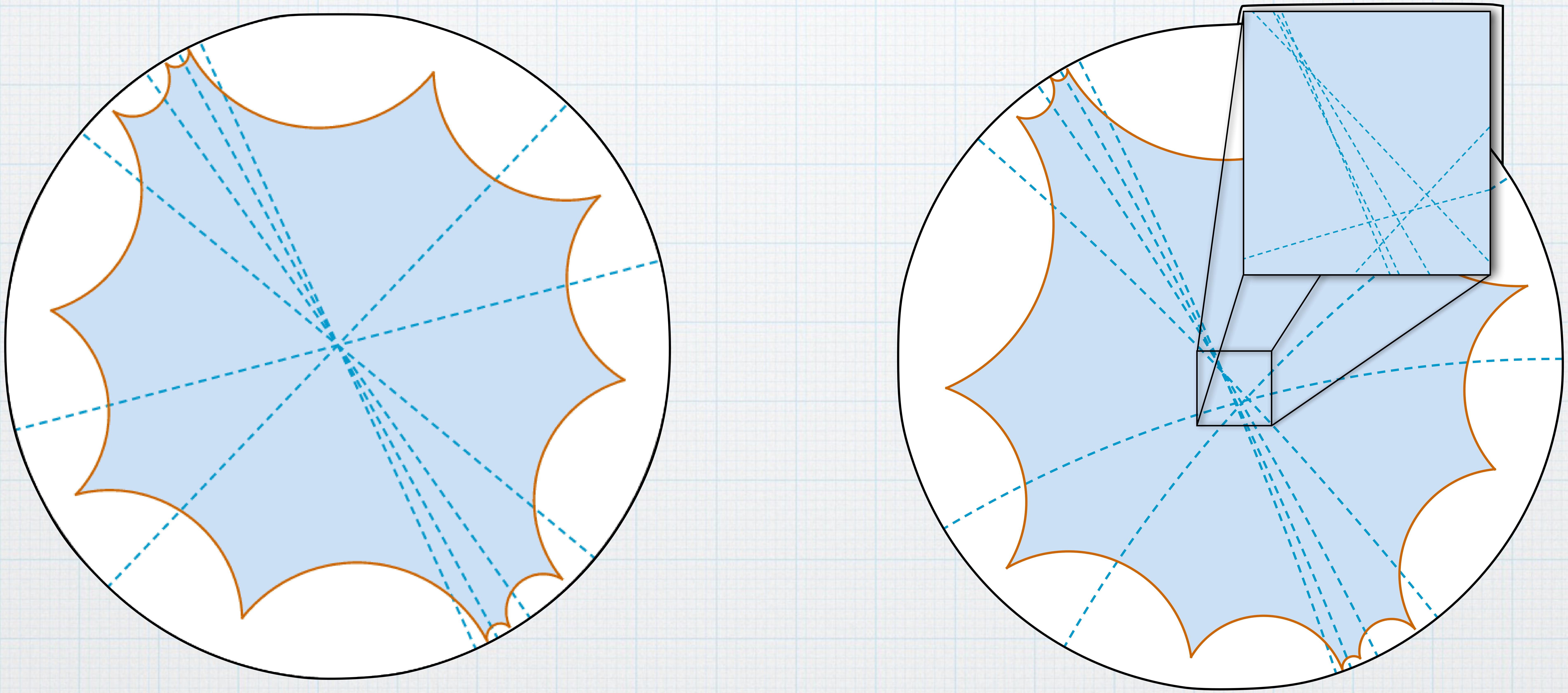
$g = 3$



$g = 4$



Non-Hyperelliptic Test



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Elliptic Curves

$$\mu^2 = \prod_{k=1}^{4} (\lambda - \lambda_k)$$

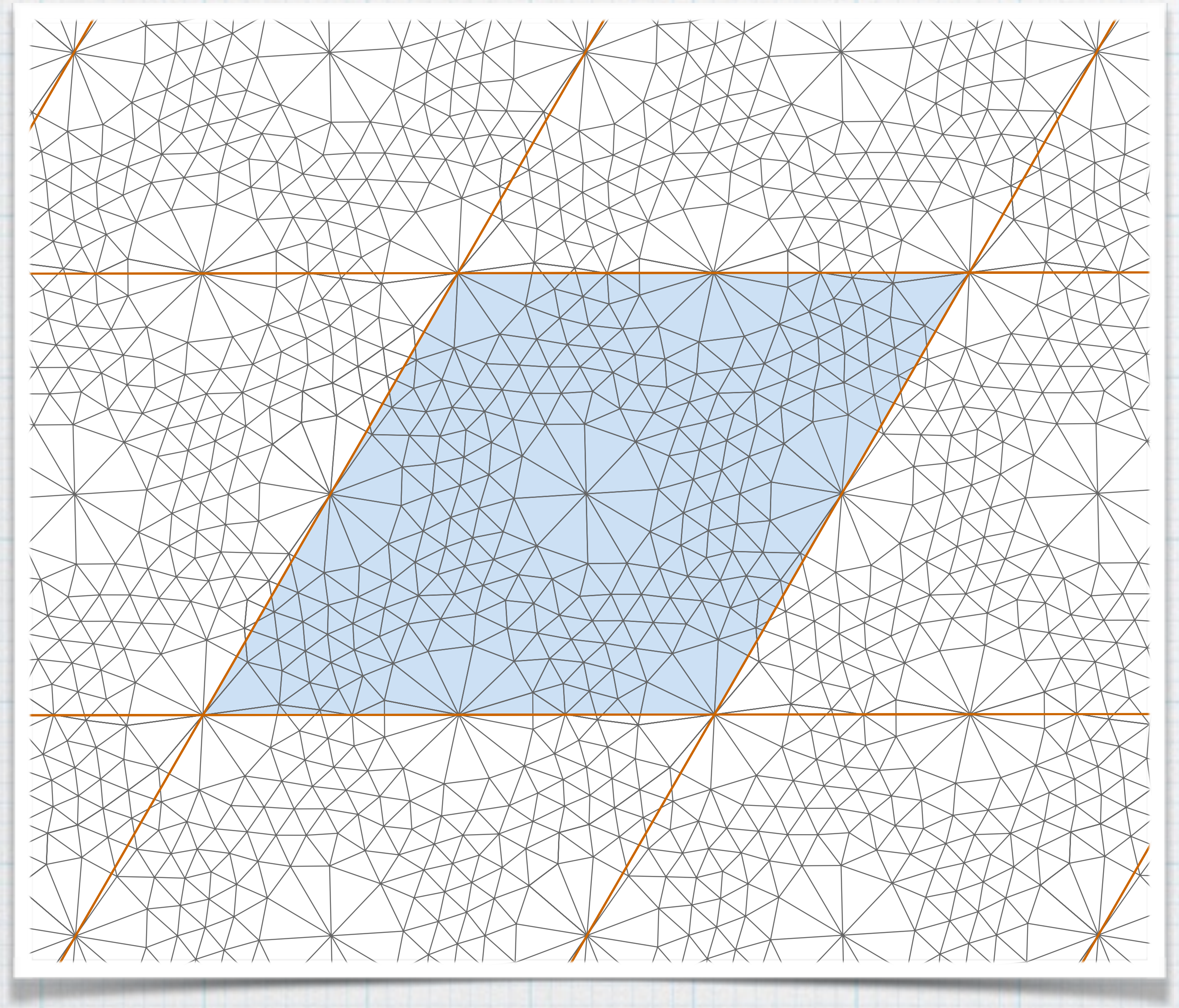
- Use representation as 2-sheeted branched cover of $\hat{\mathbb{C}}$
- Conformal invariant τ can be calculated to high precision from $\lambda_1, \ldots, \lambda_4$
- Use τ as ground truth in convergence comparisons

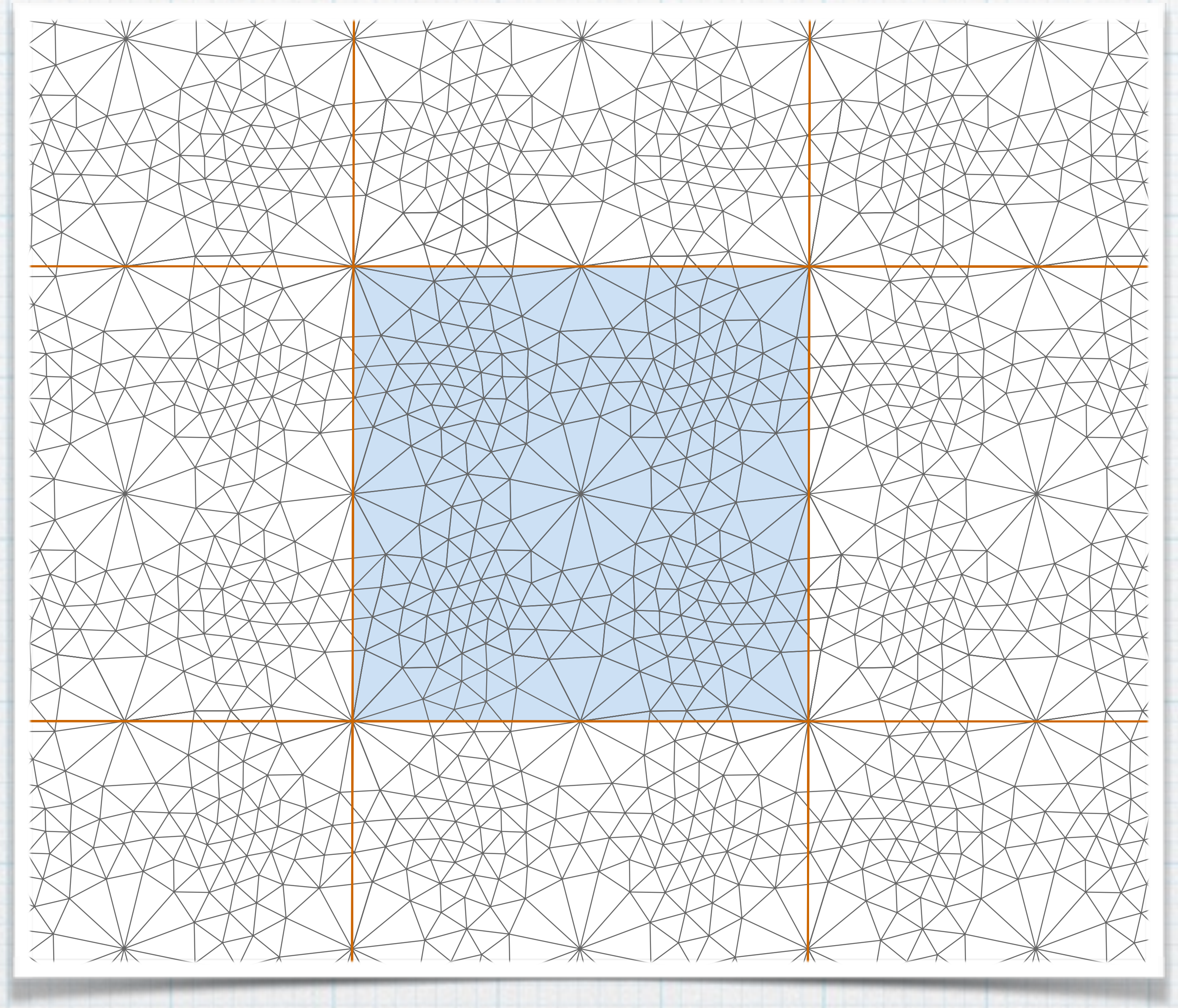
Example

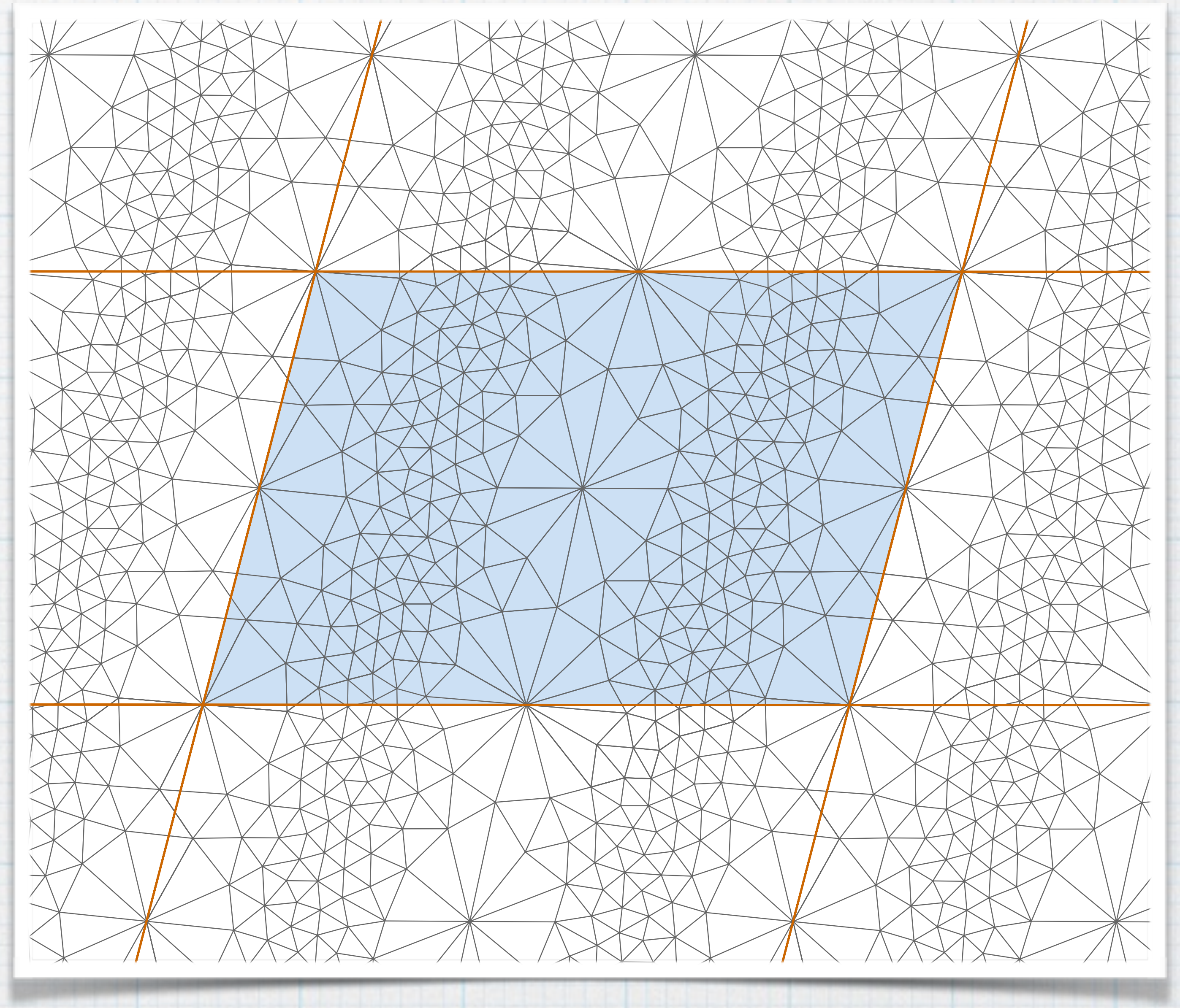
$$\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\tau = i$$

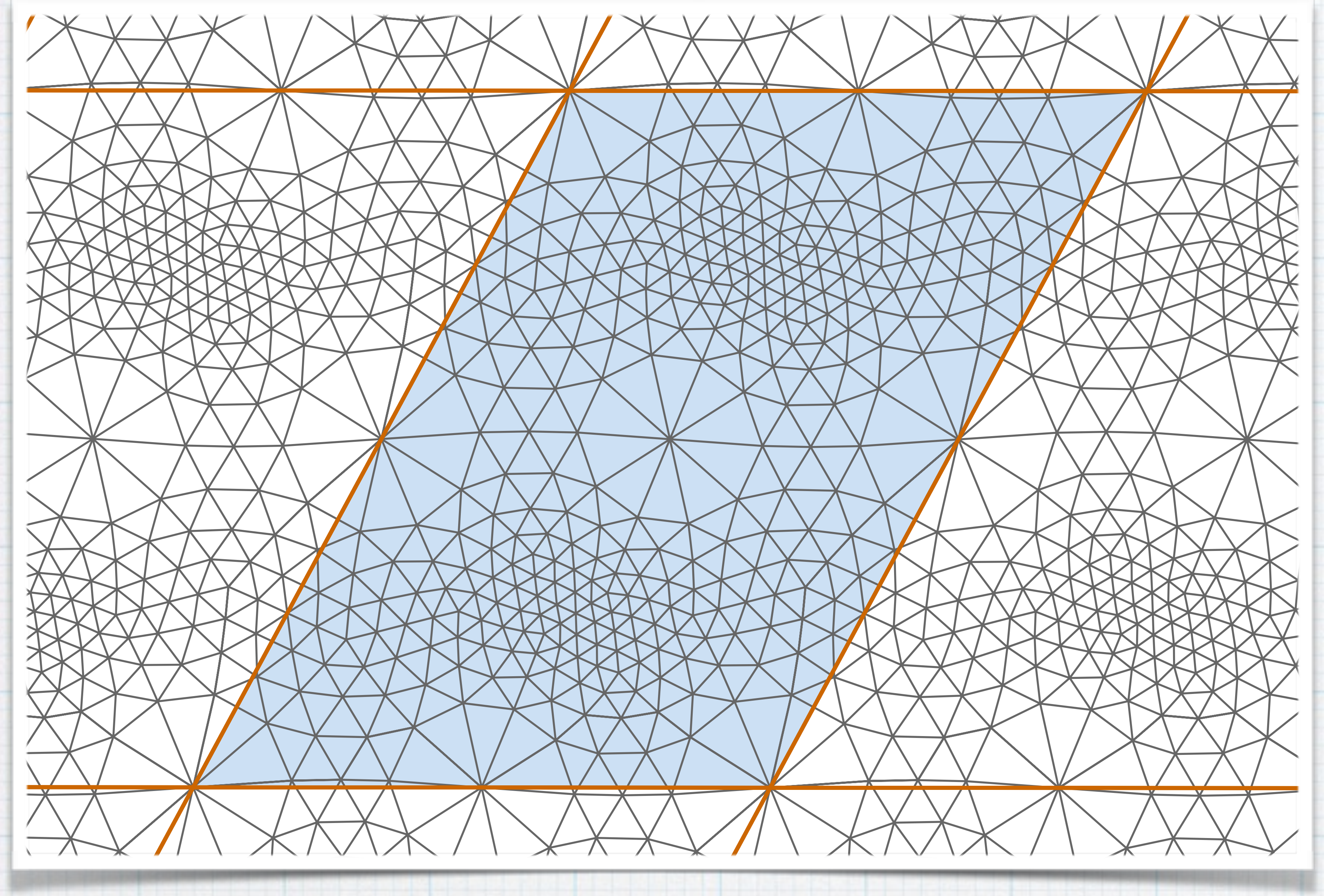
$$\tau = z$$

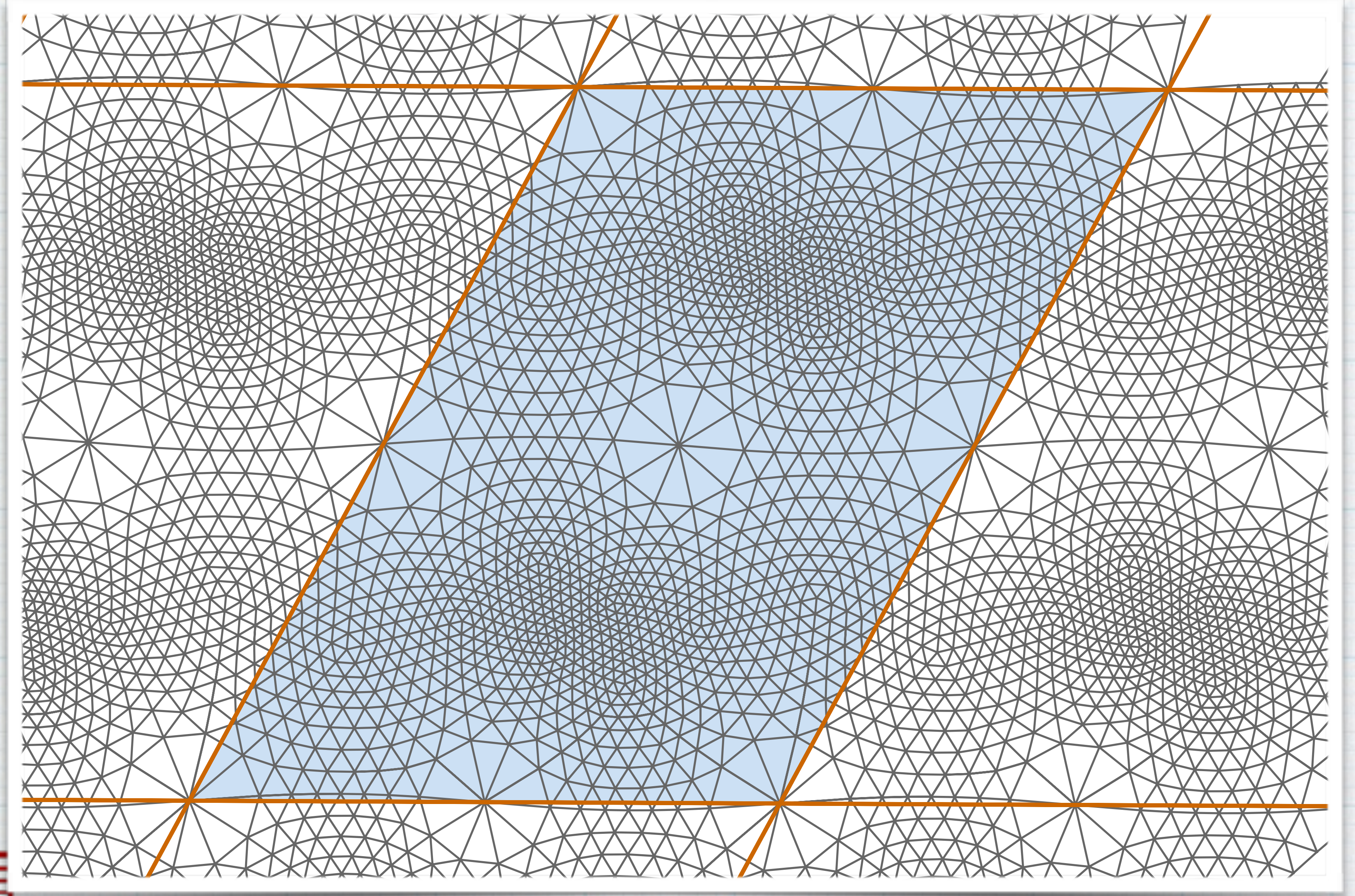


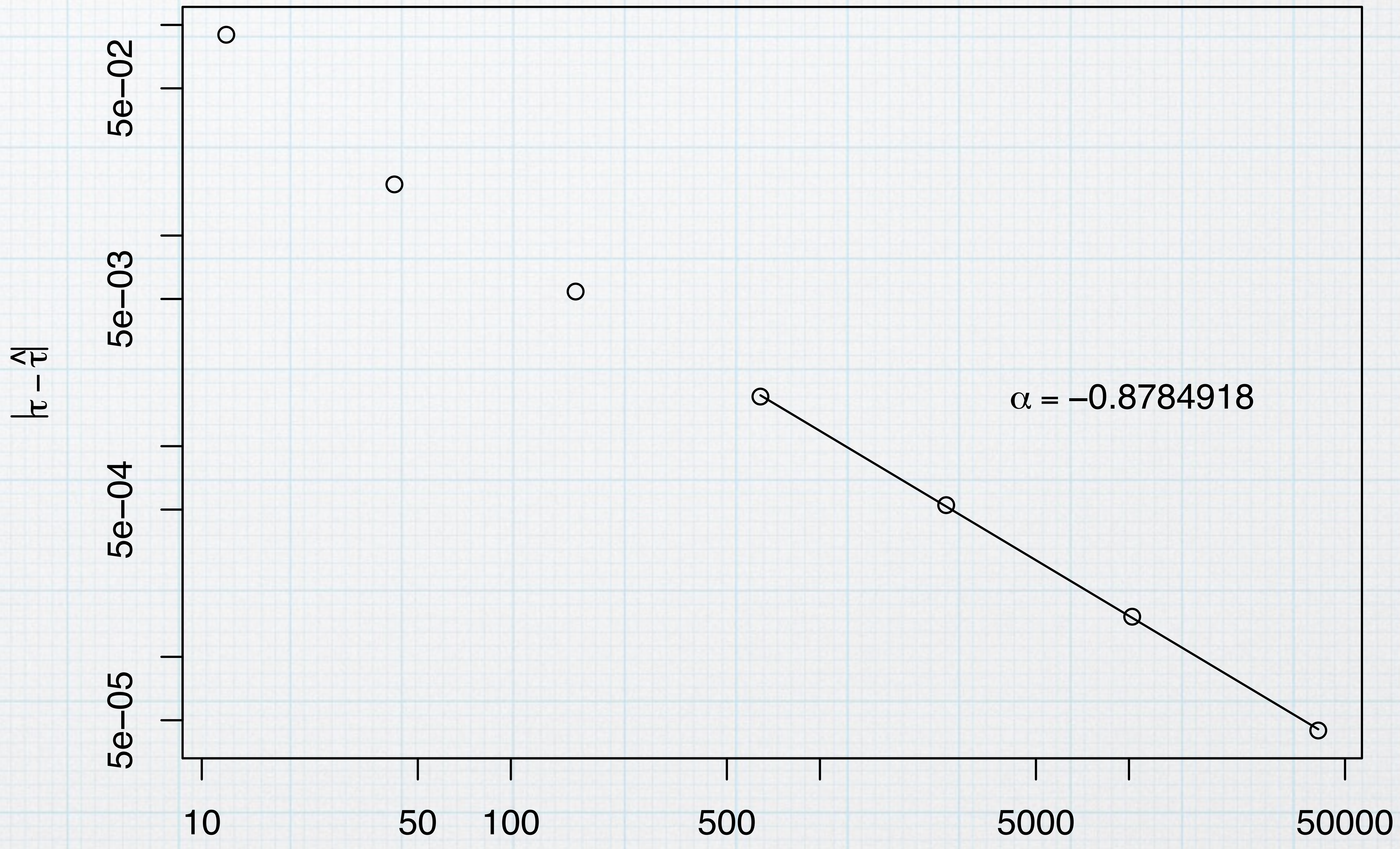




Convergence

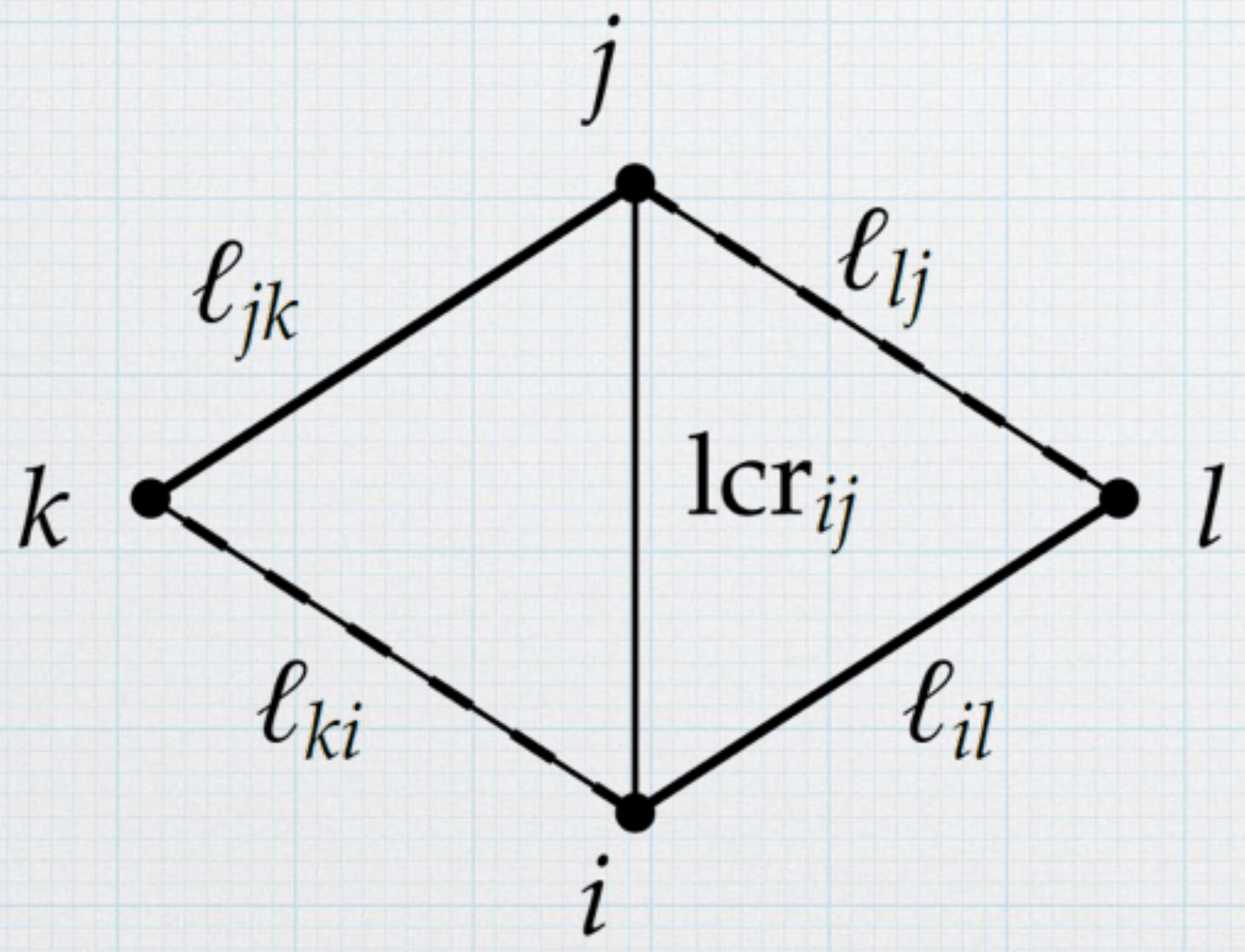




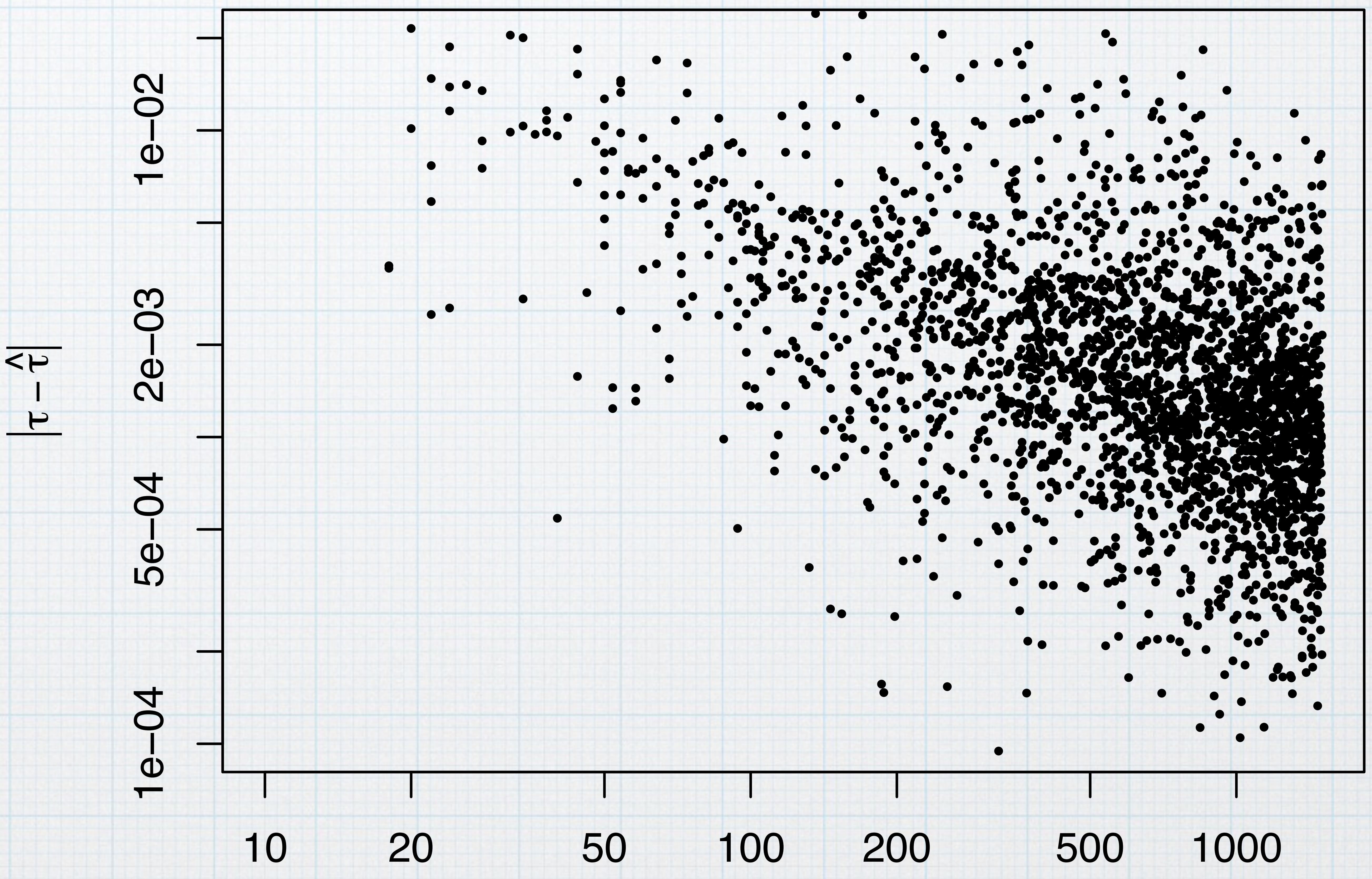


Convergence Behavior

$$Q_{ij} := \frac{1}{2}\left(\mathrm{lcr}_{ij} + \frac{1}{\mathrm{lcr}_{ij}}\right) - 1$$



Random Points



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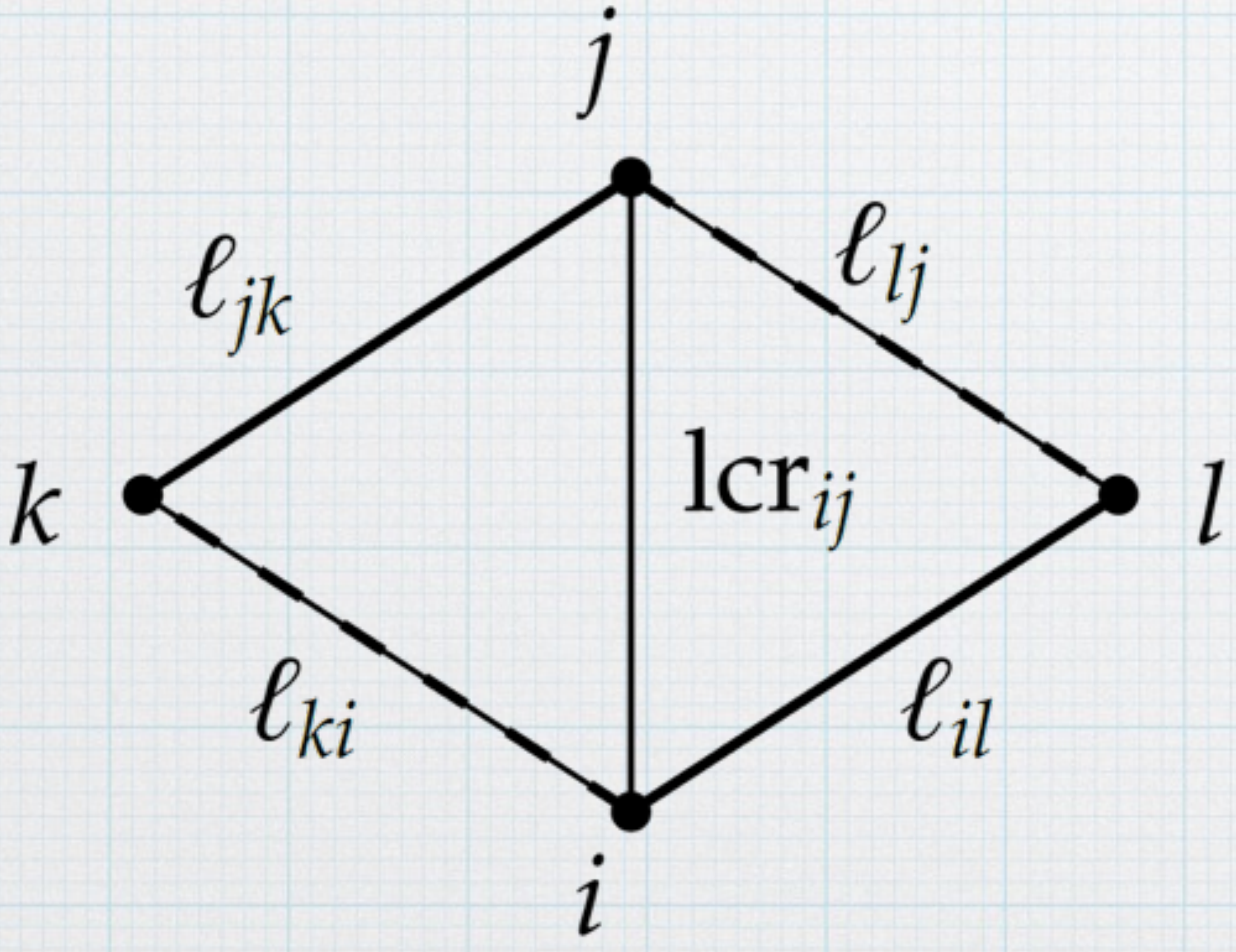
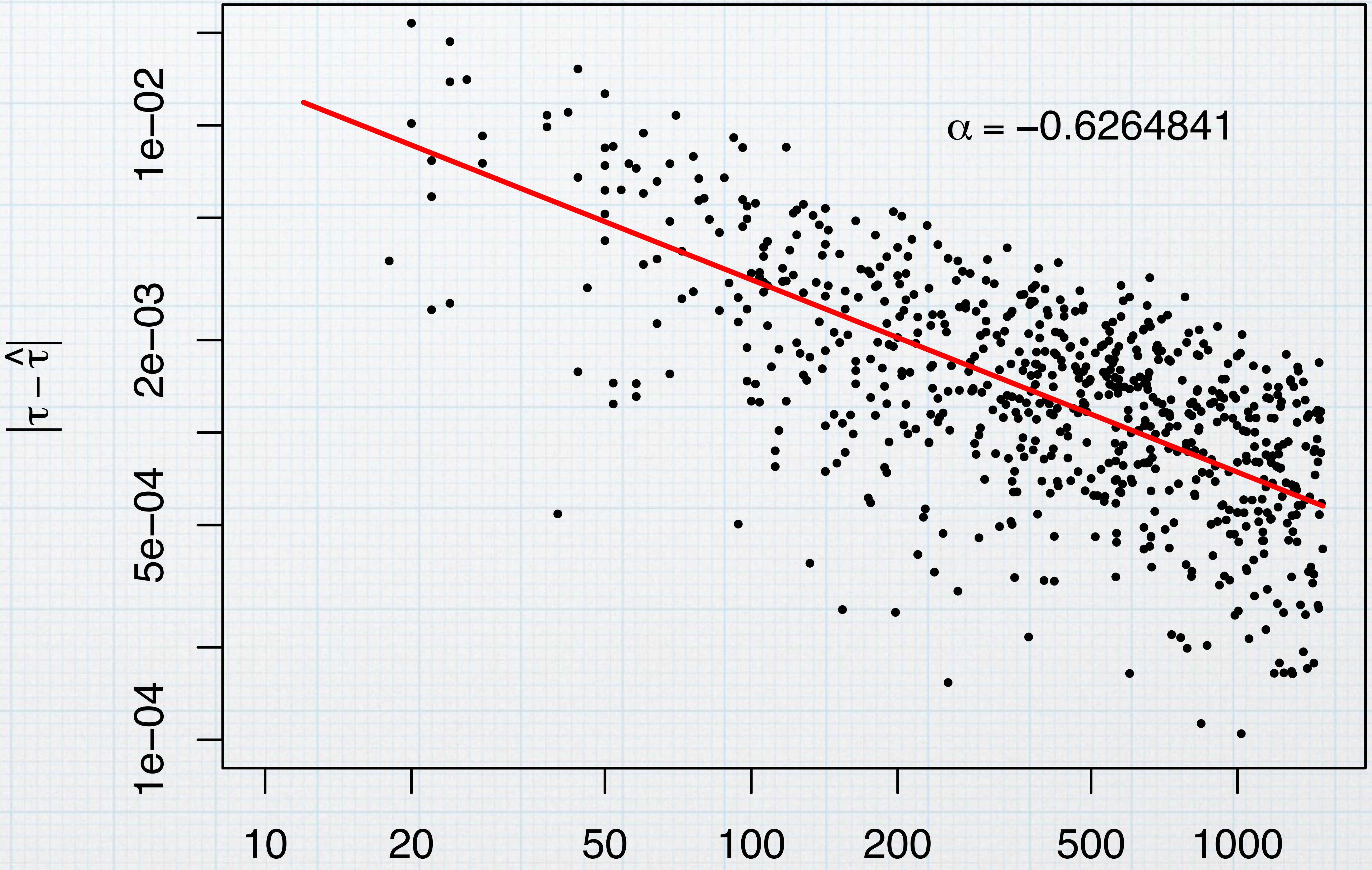


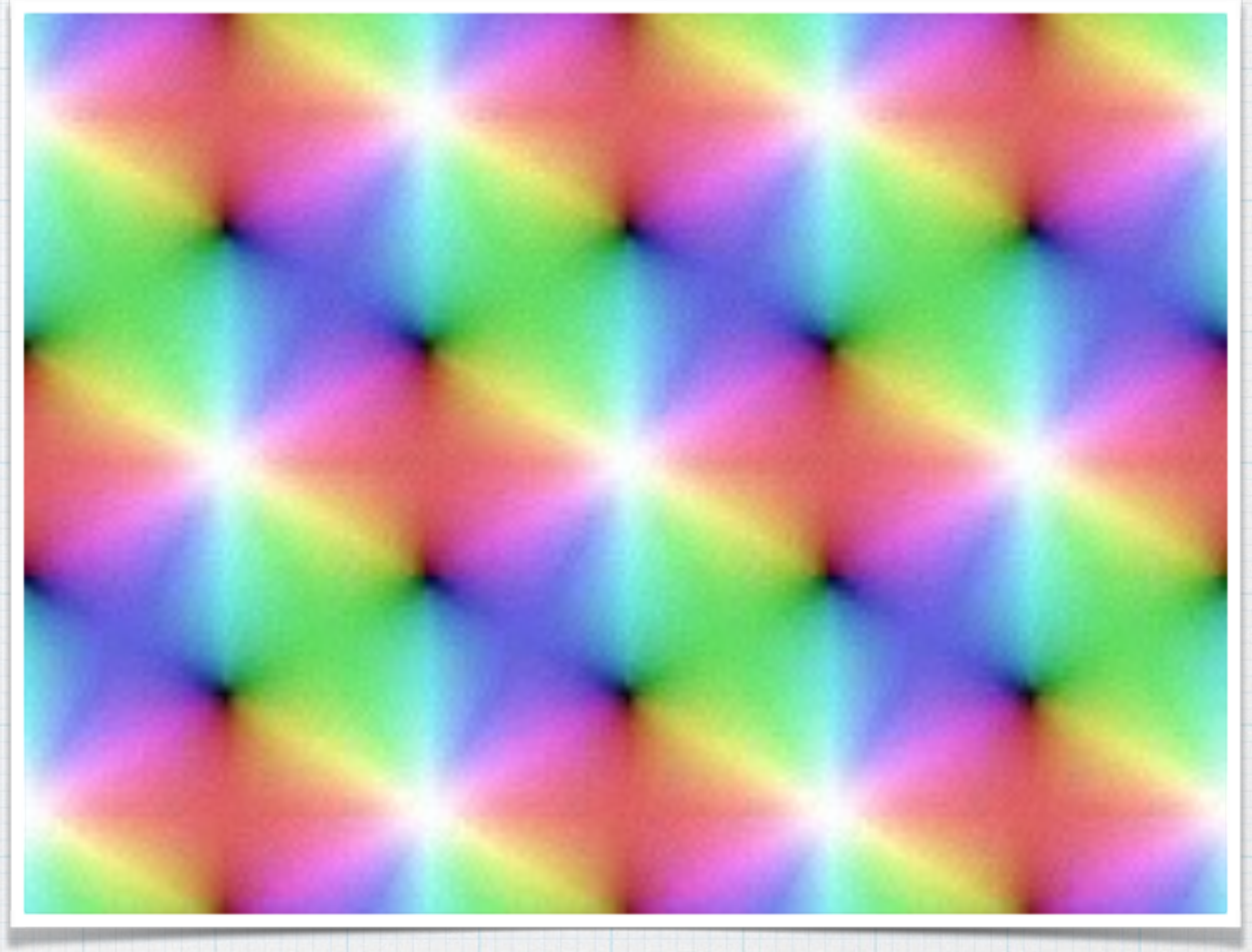
Filtered by Mesh-Quality



Elliptic Functions

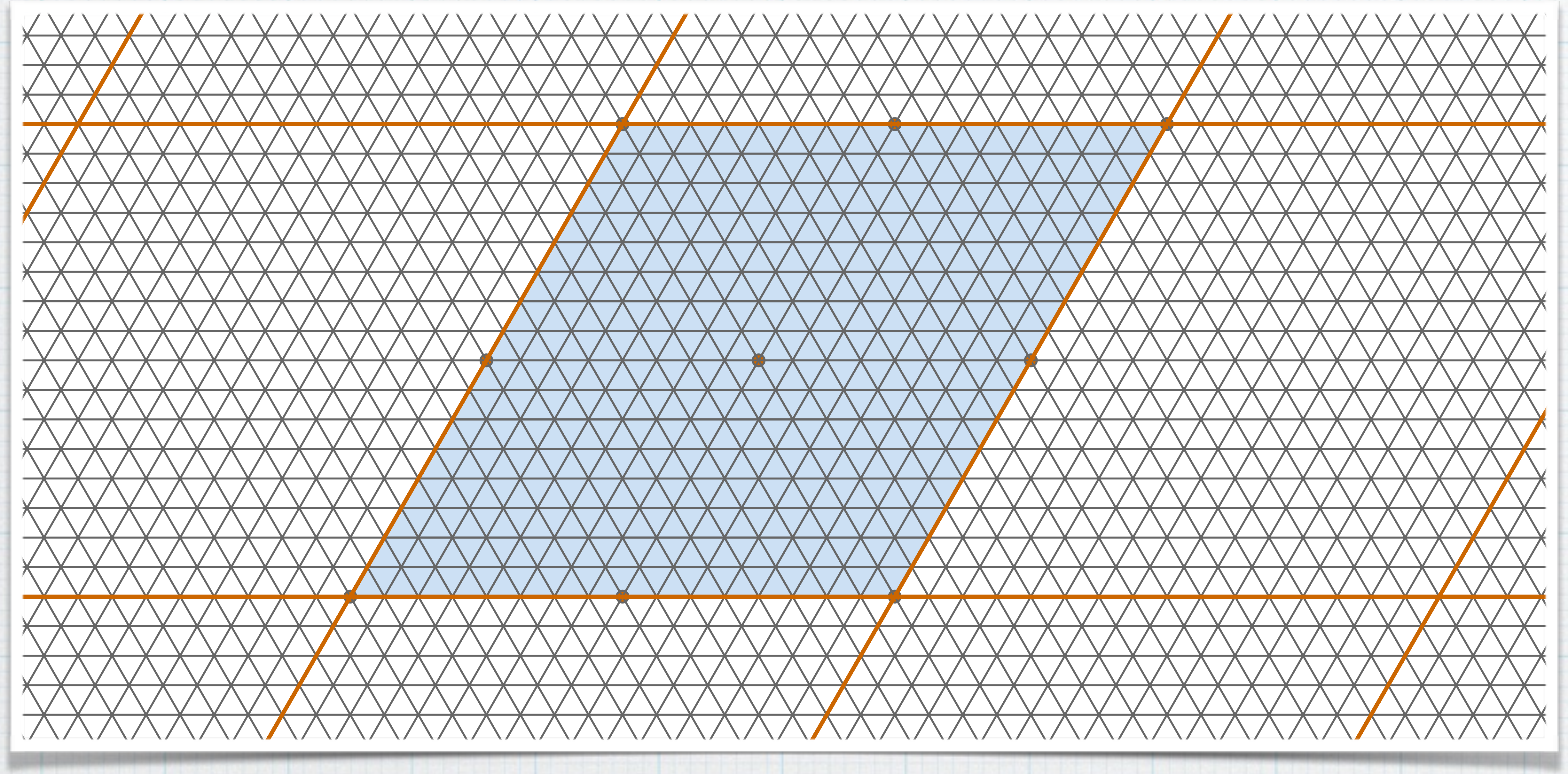
$$\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

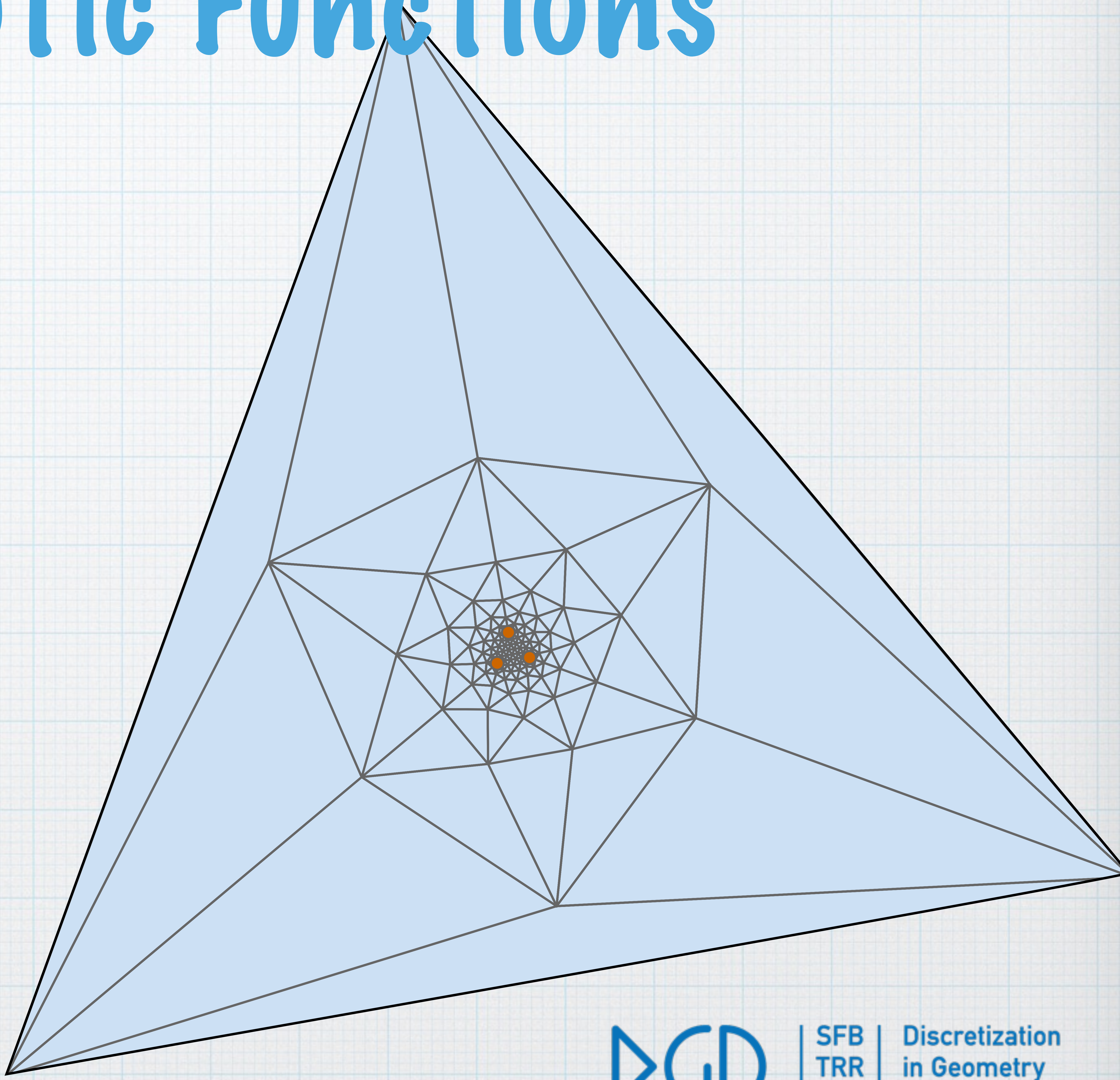
$$\wp(z) = \frac{1}{z^2} + \sum_{\gamma \in \Gamma \setminus \{0\}} \left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right)$$



Discrete Elliptic Functions

$$\Gamma = \mathbb{Z} + \tau\mathbb{Z} \qquad \tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$





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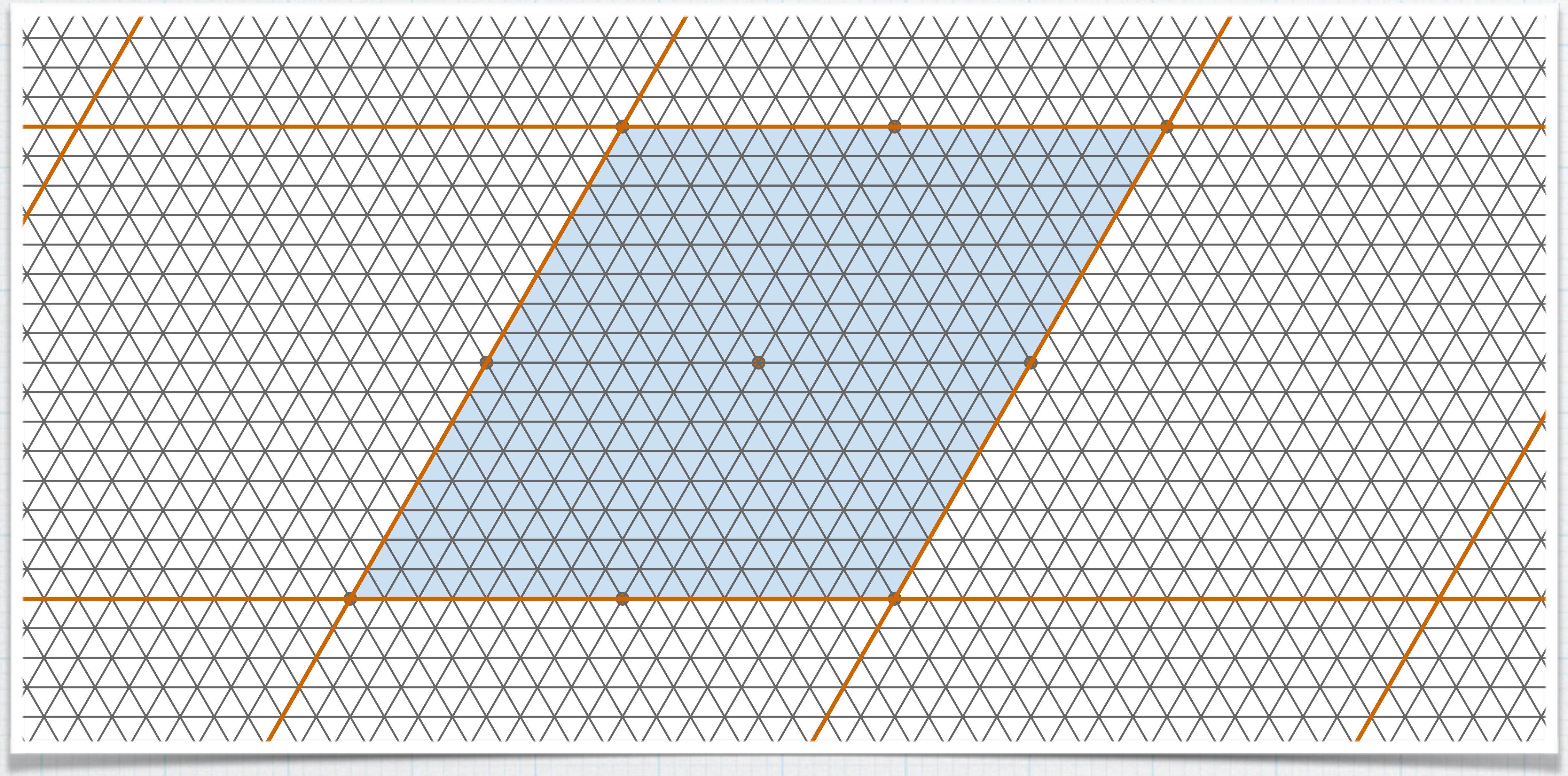
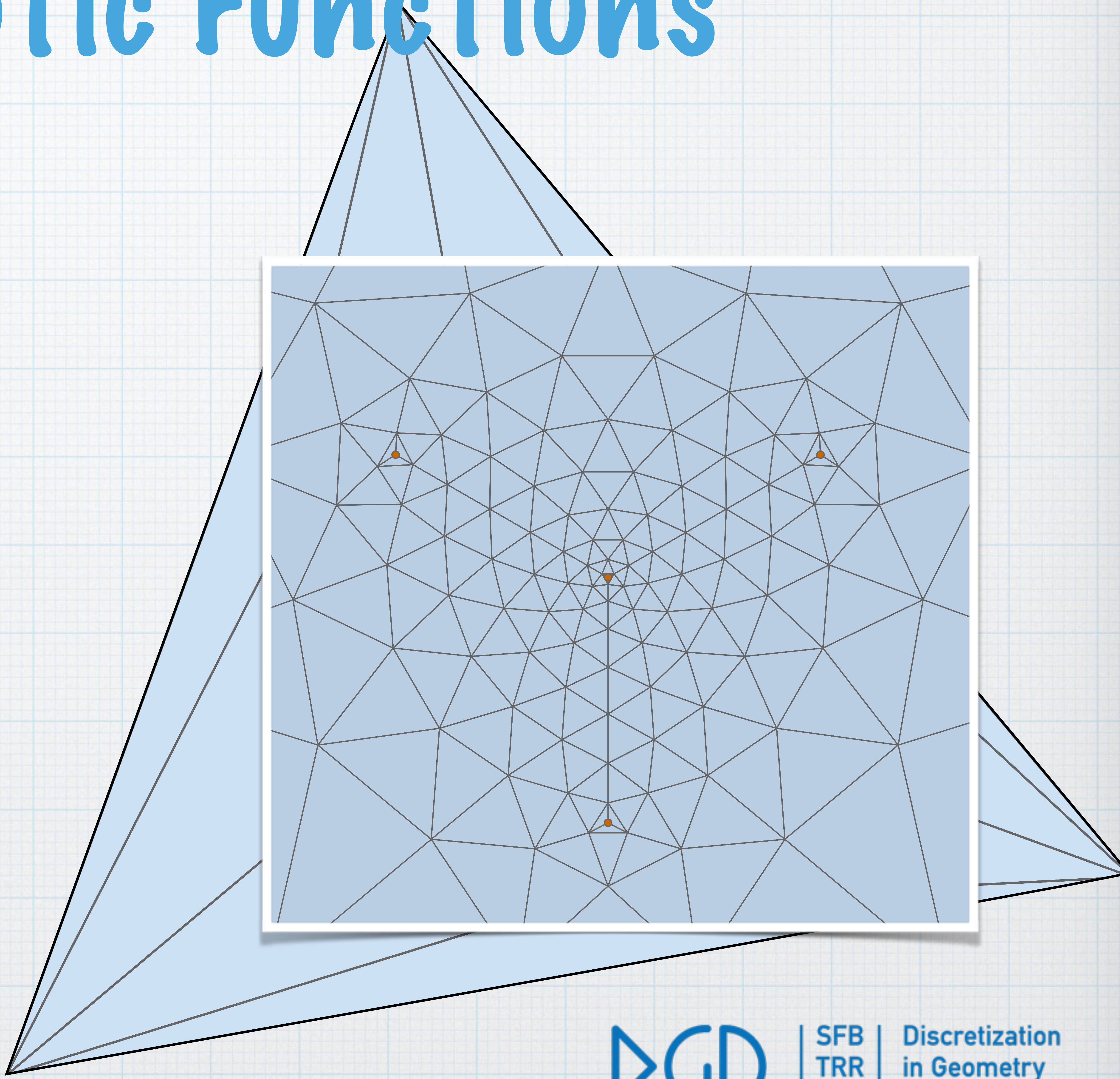
Discrete Elliptic Functions

$\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ $\qquad$ $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$





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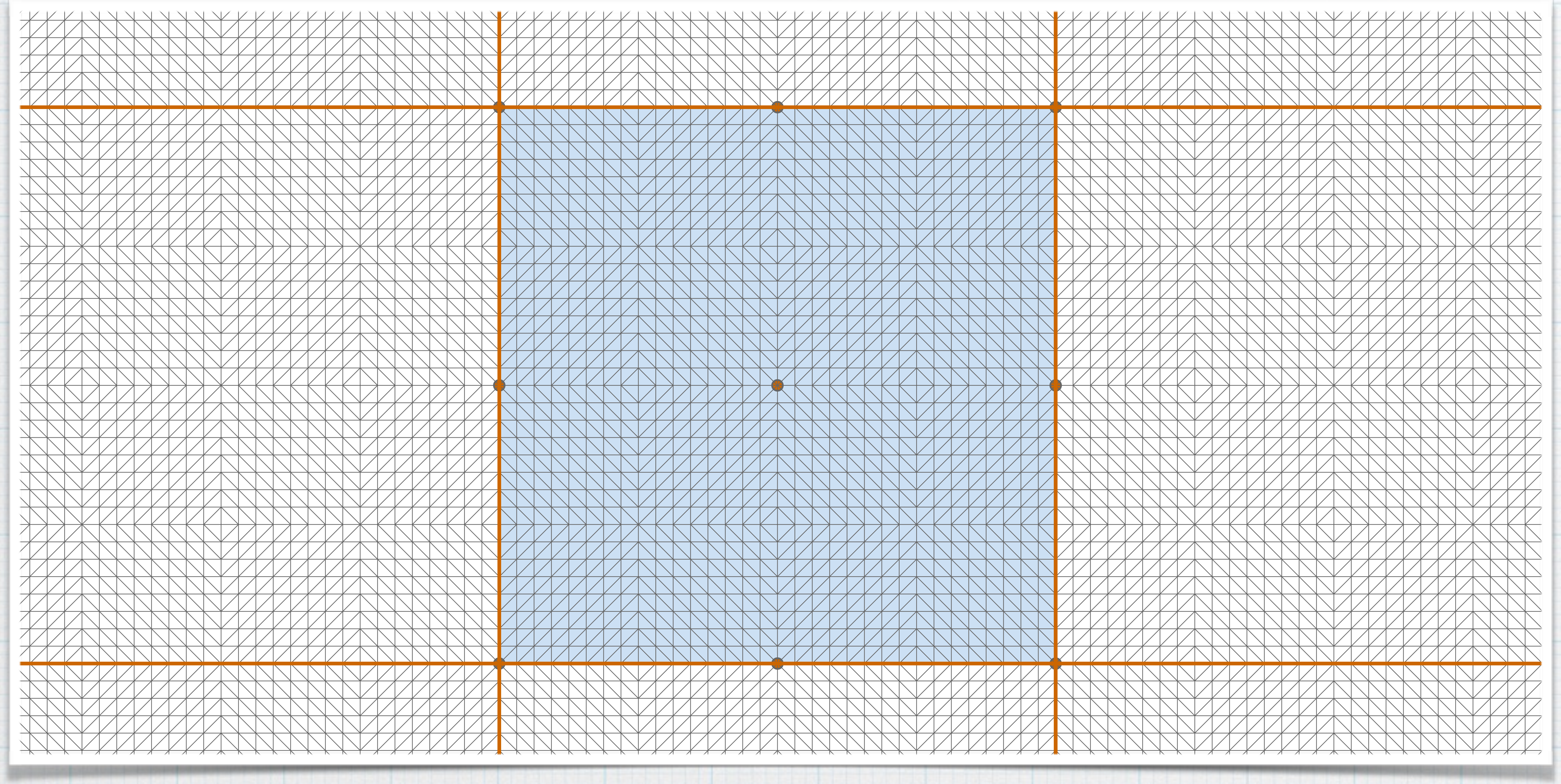
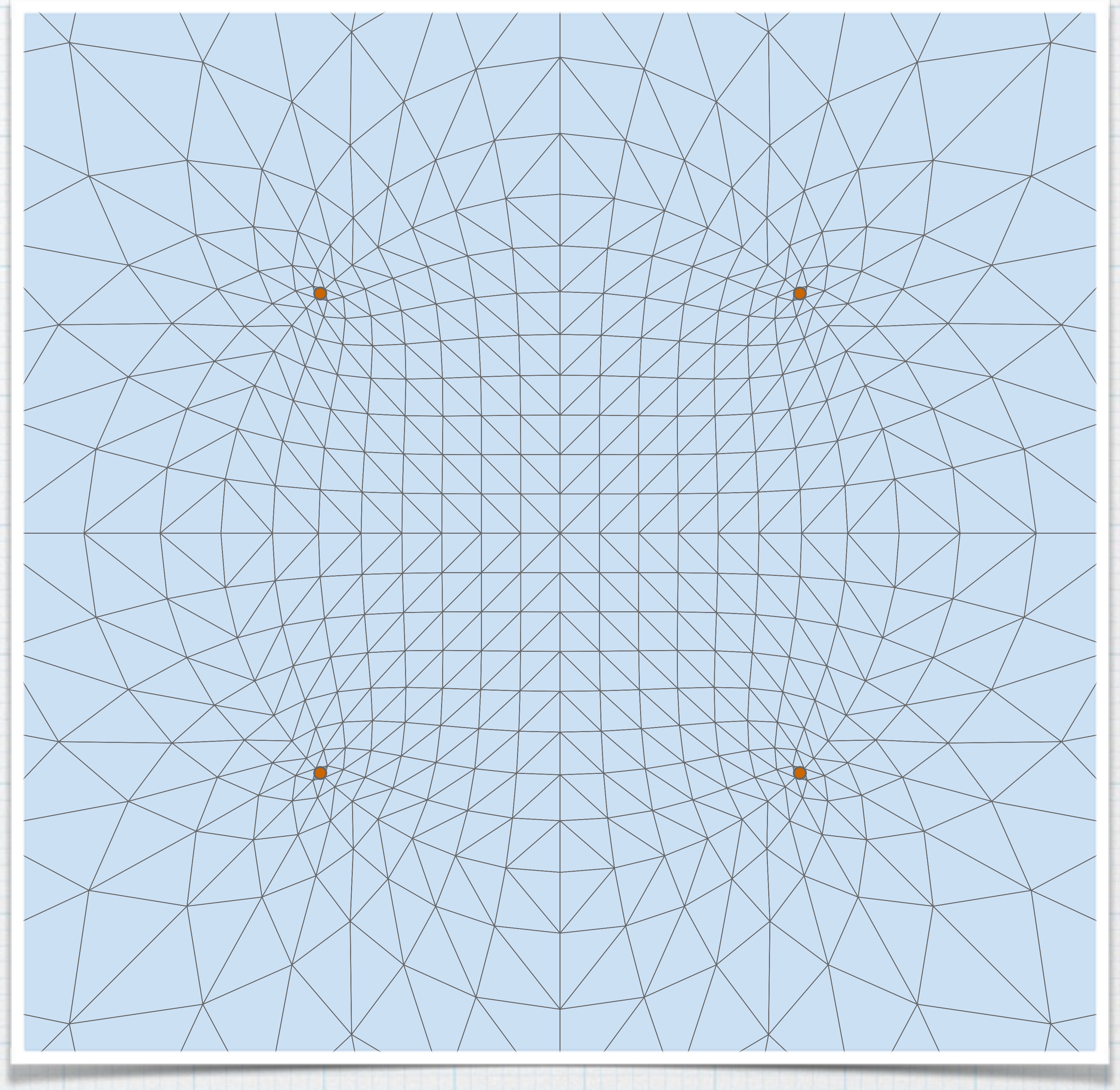
Discrete Elliptic Functions

$\tau = i$





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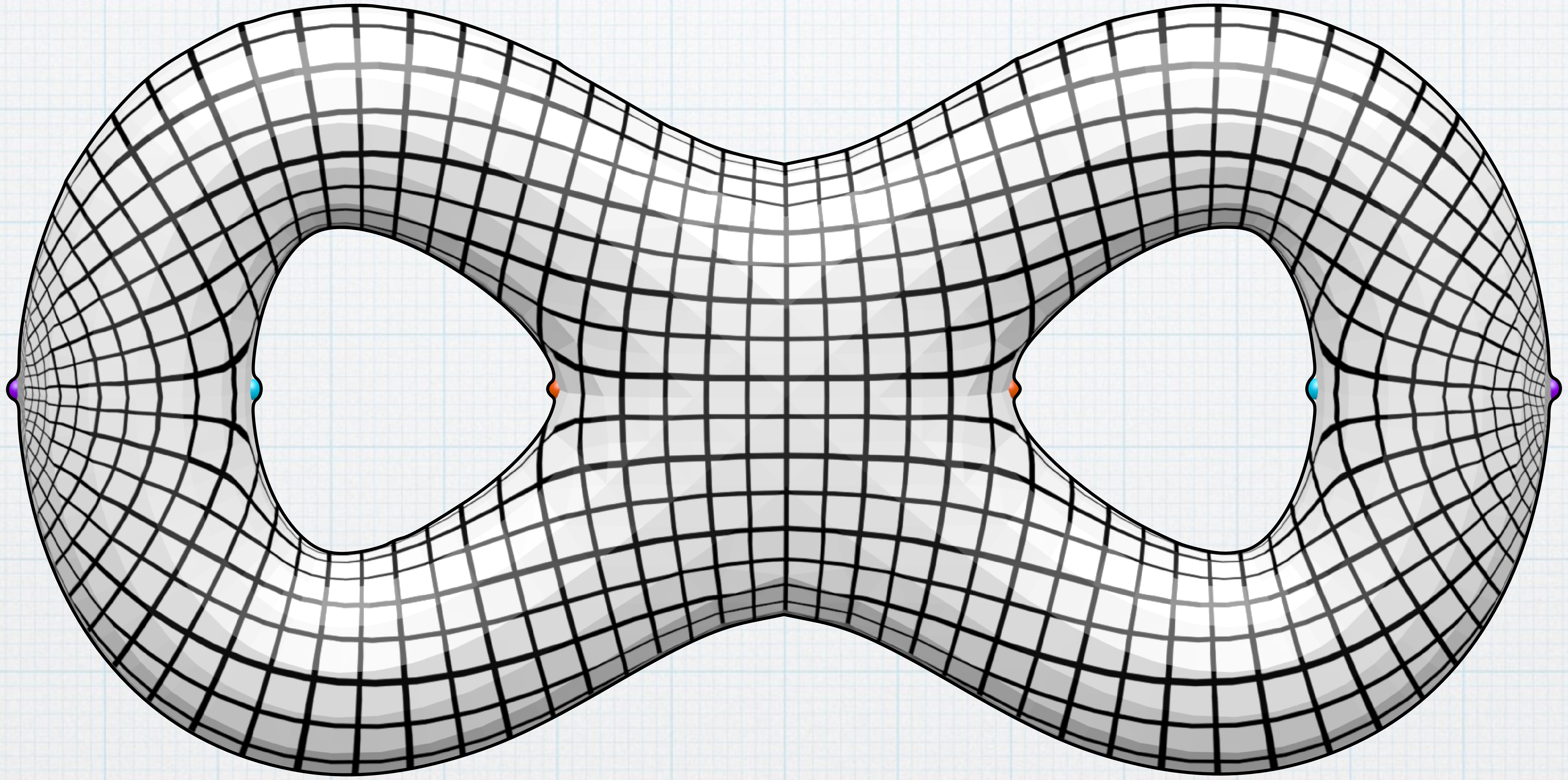
Inverse Mapping



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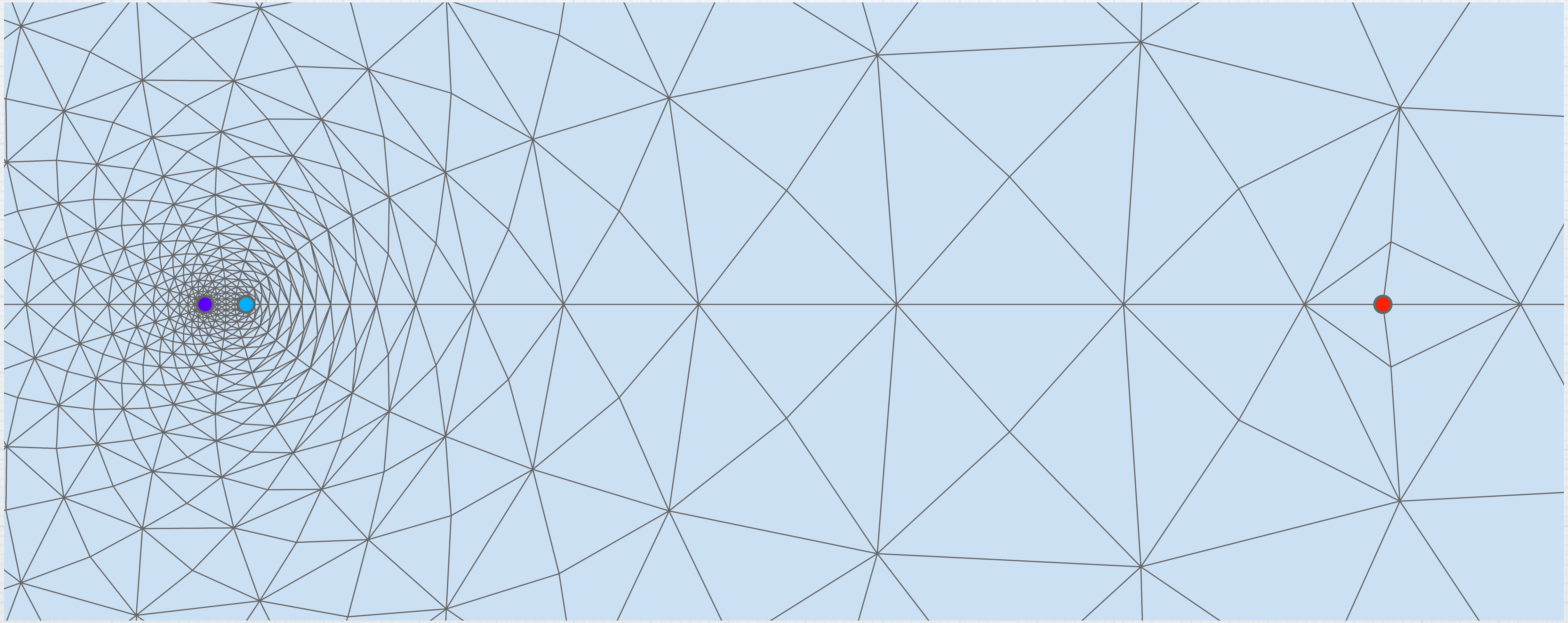

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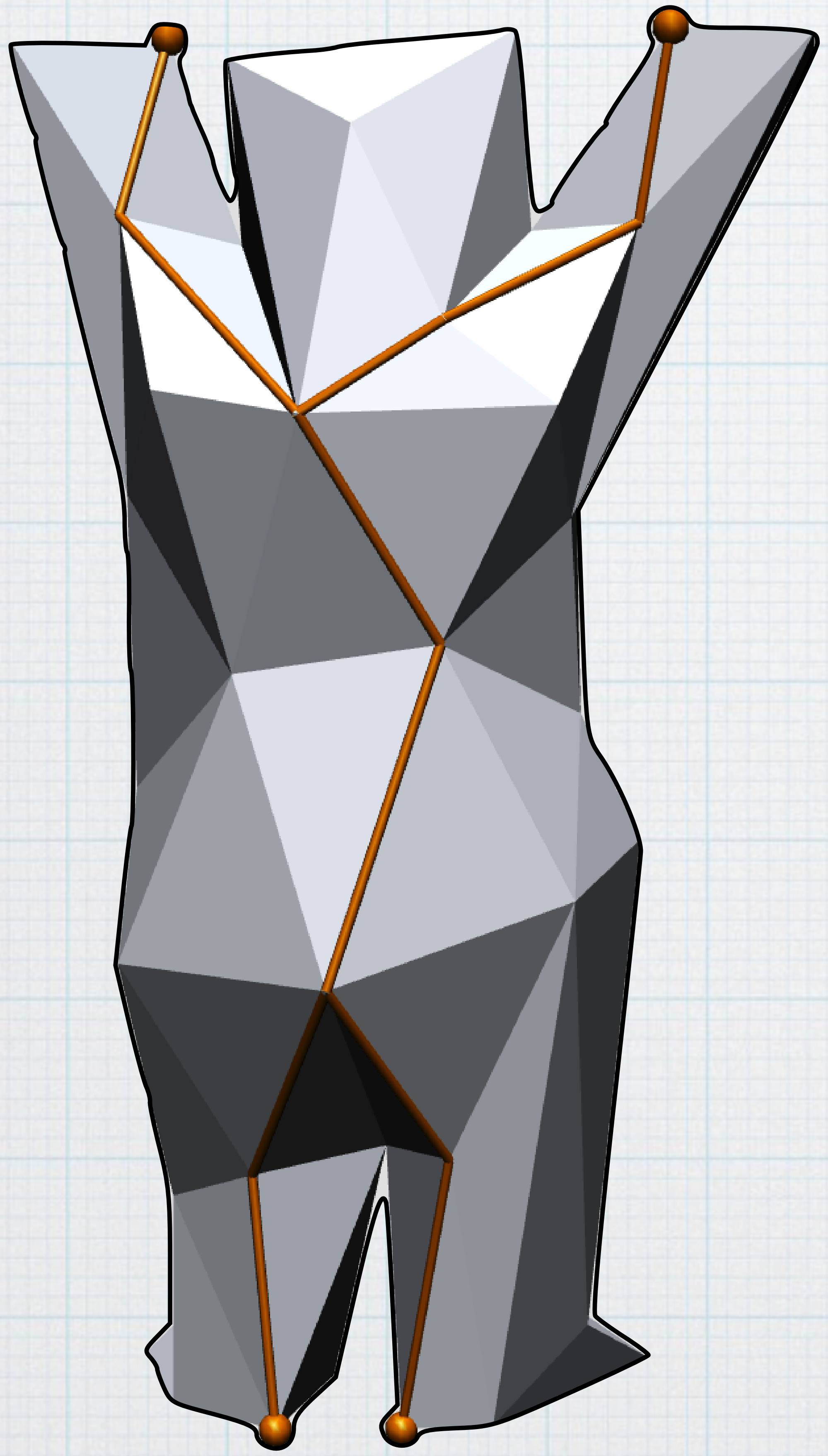
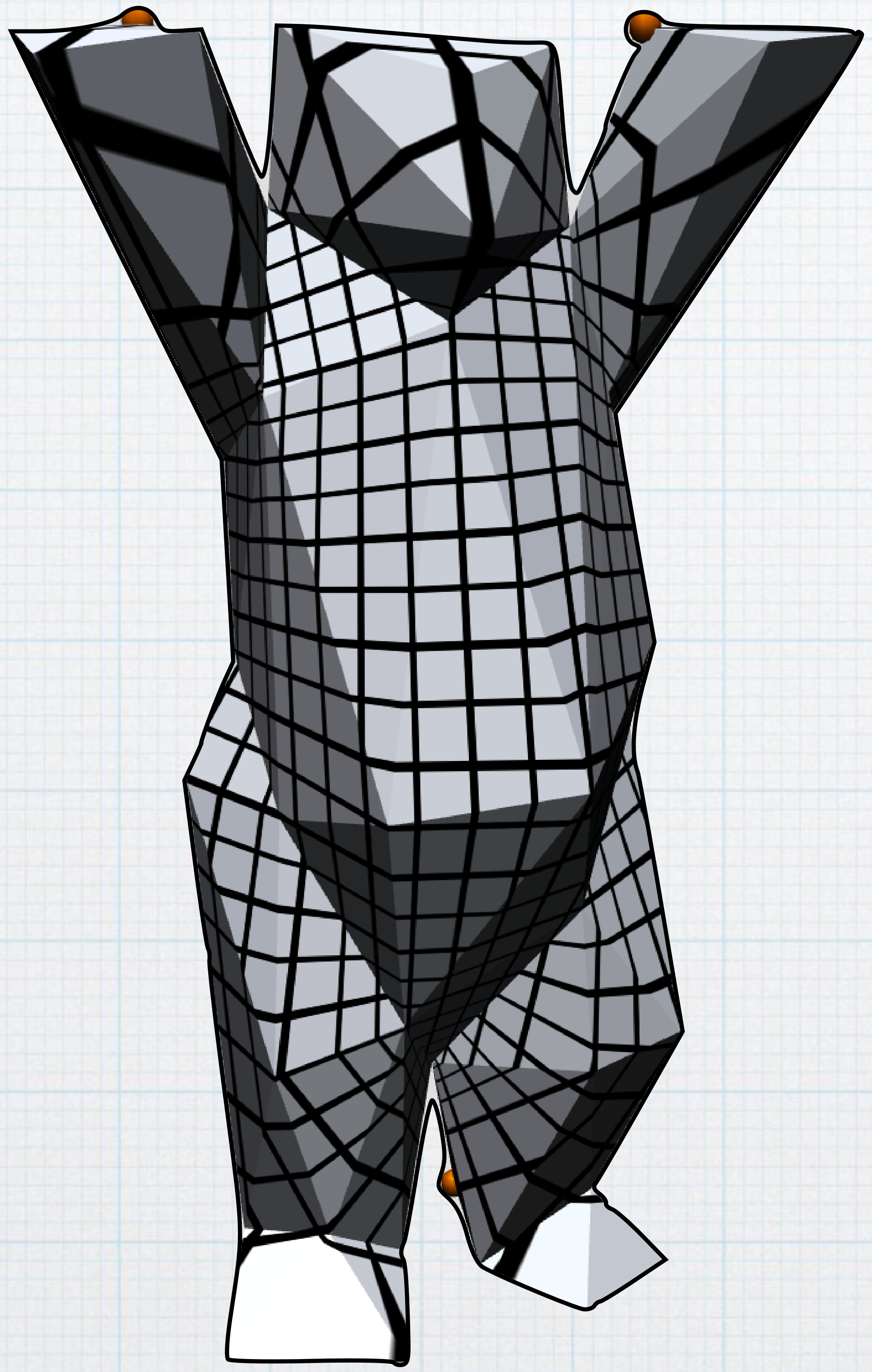
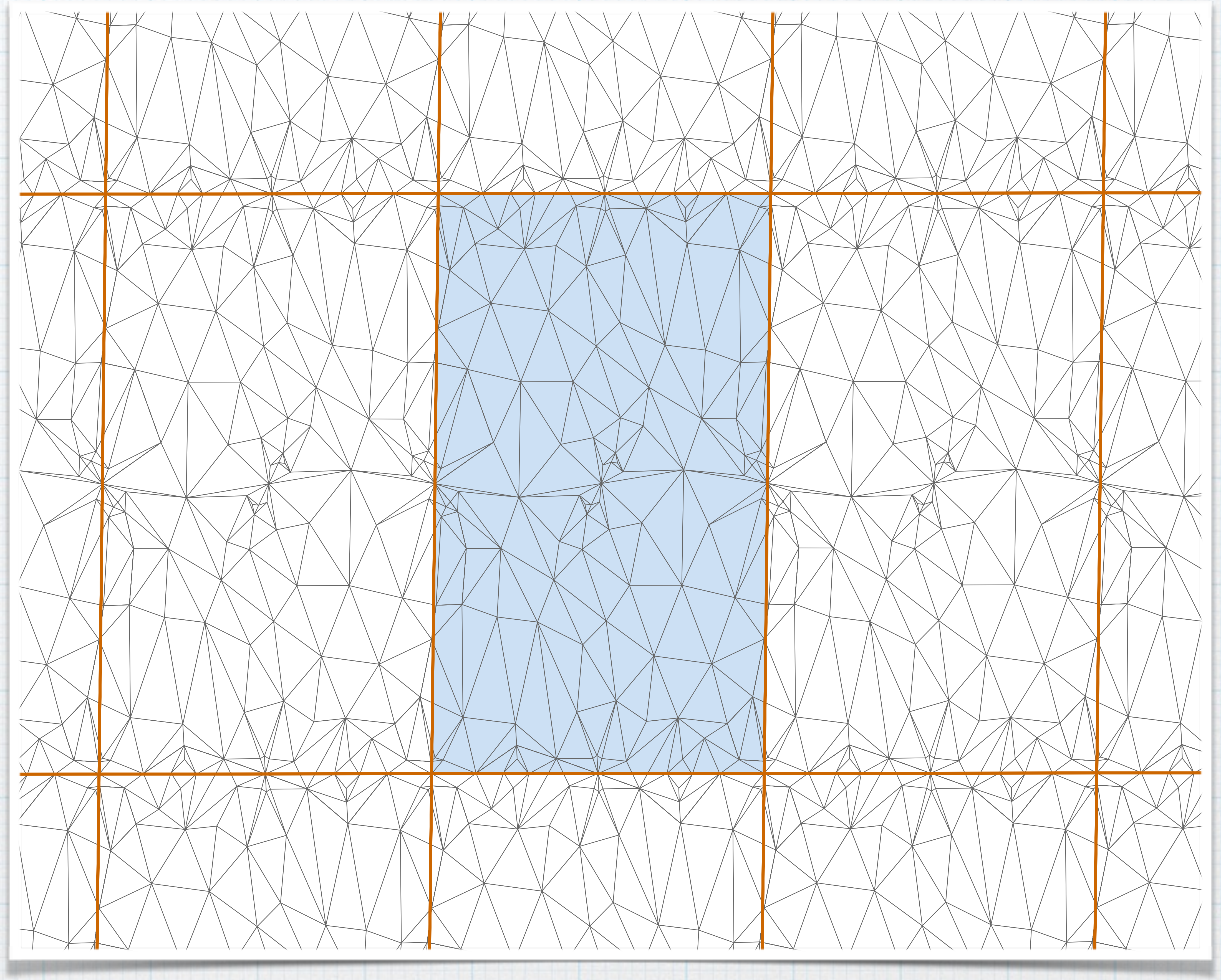
Euclidean Spheres







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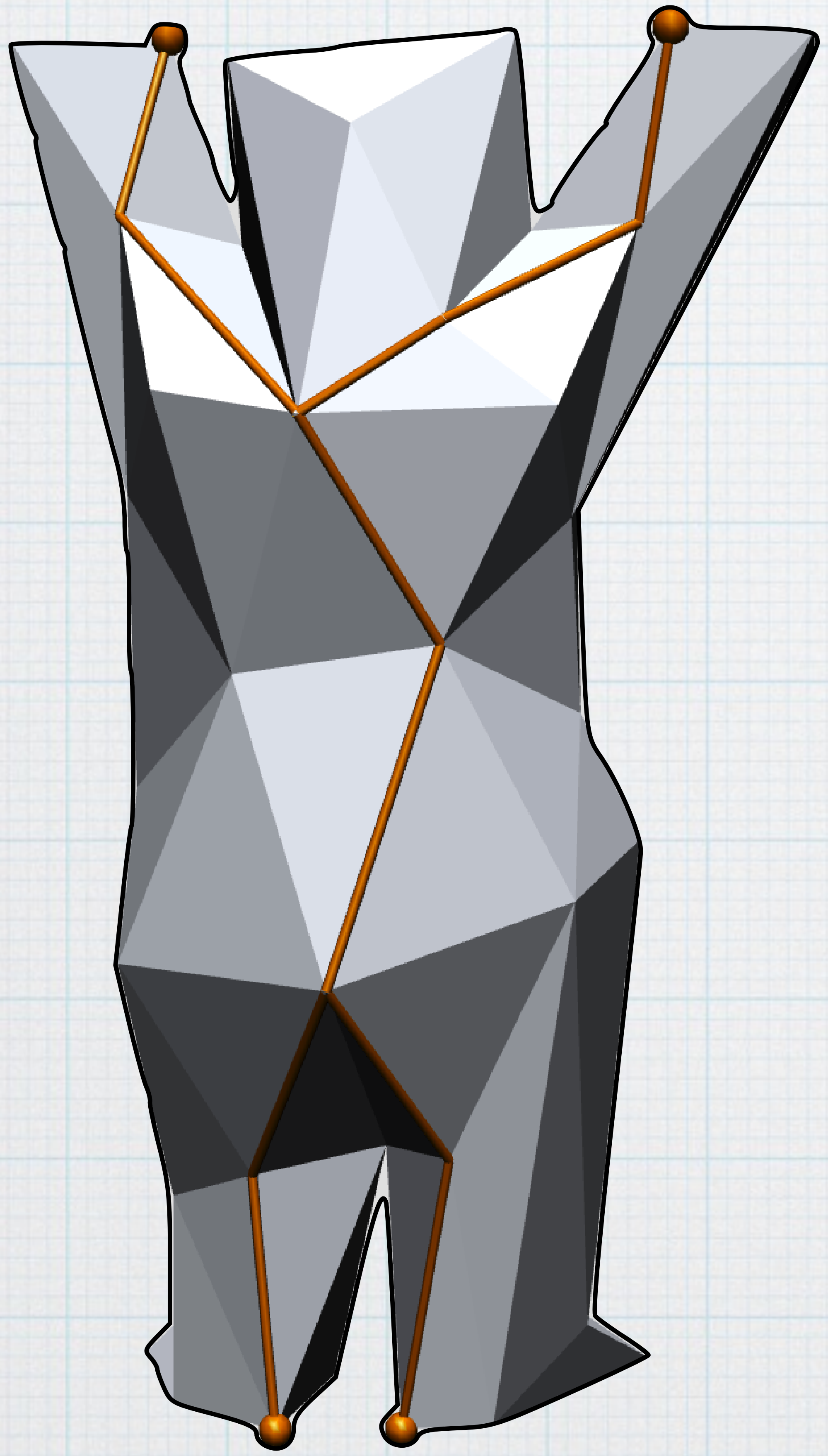

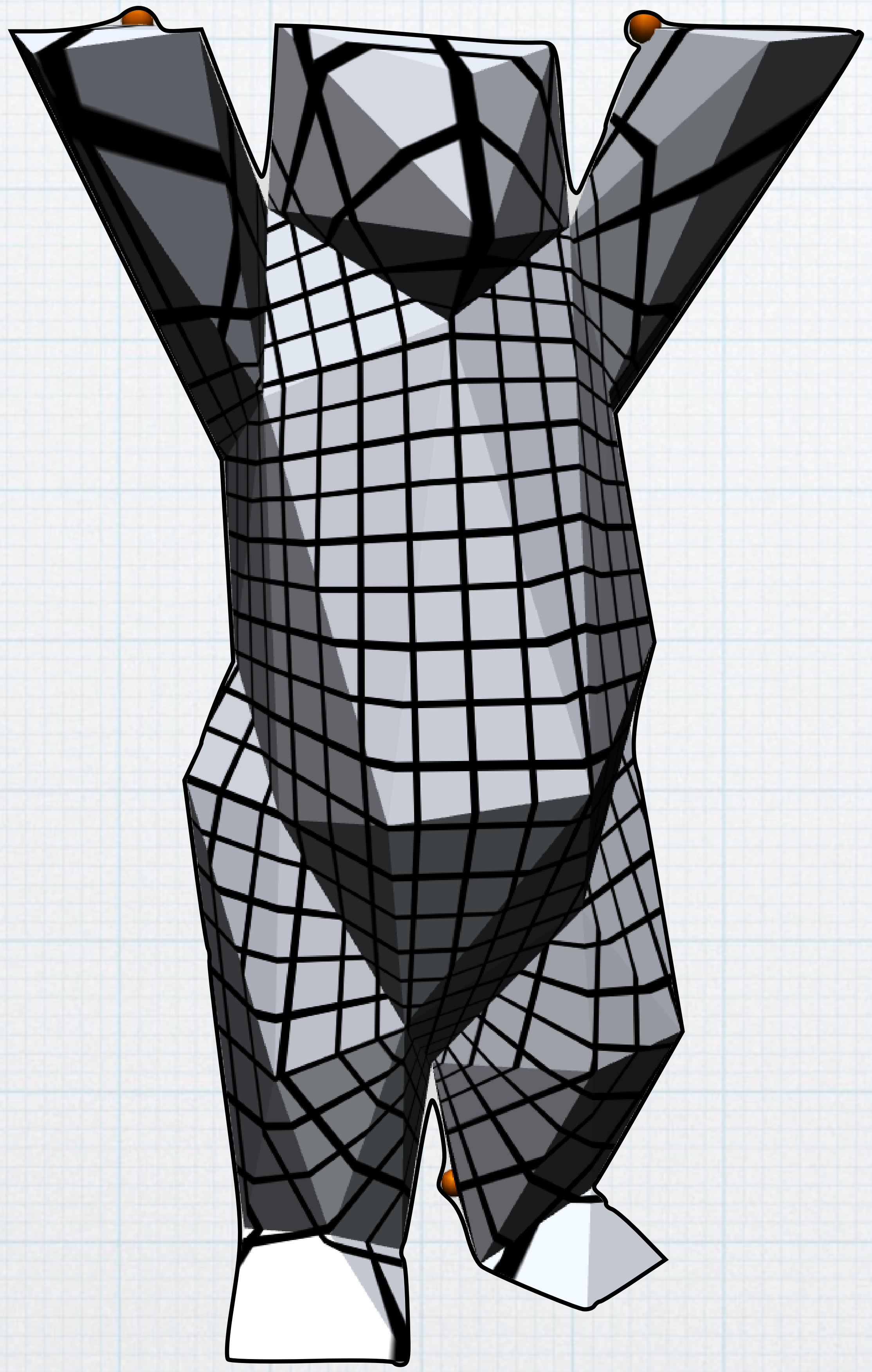

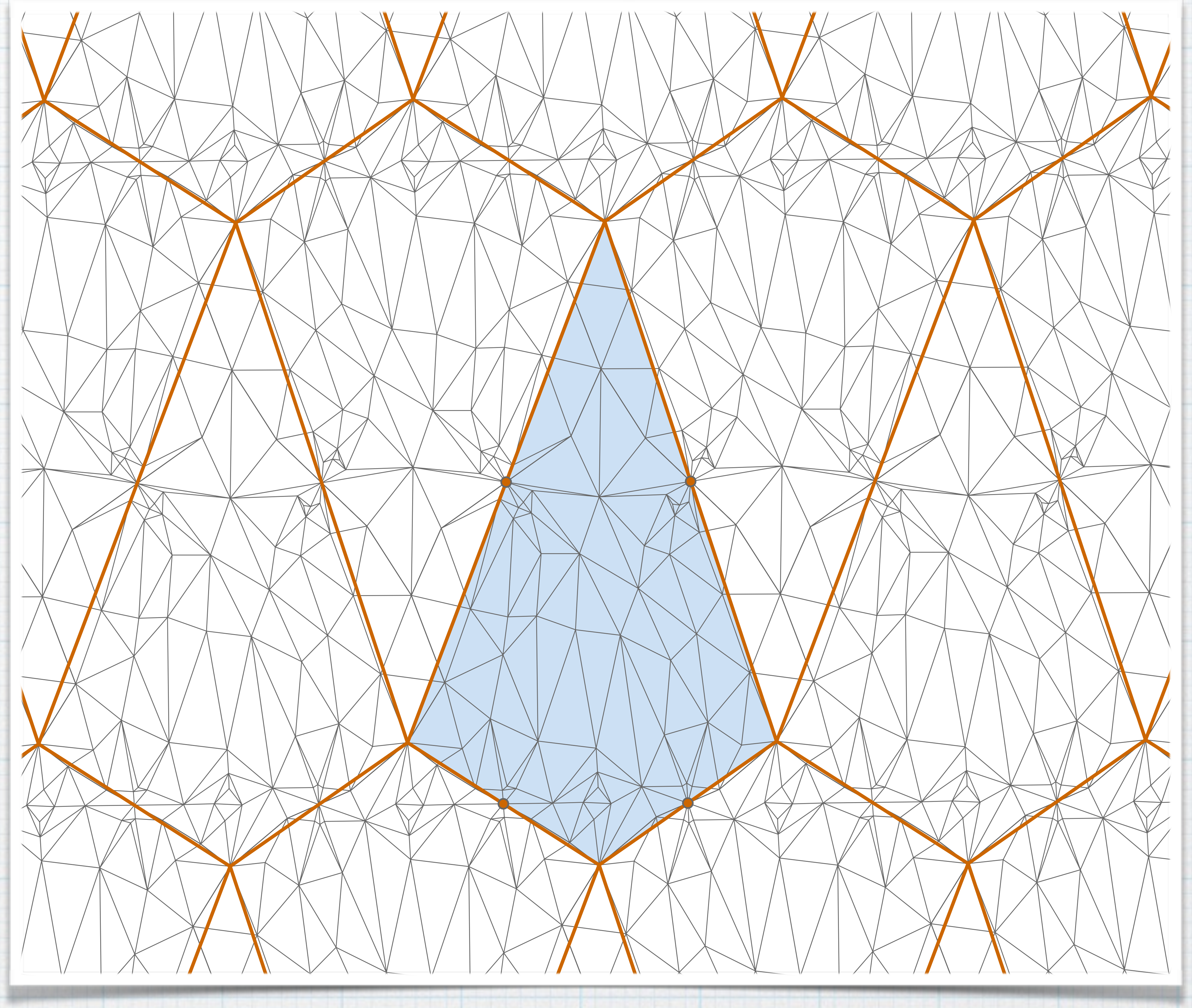

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